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Representation results for integral and supremal functionals

Elvira Zappale

Università di Salerno

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Topics

Integral functionals

$$\begin{cases} \text{Homogenization} \\ \text{Integral Representation} \\ \text{Dimensional Reduction} \end{cases}$$

related to imaging, elasticity and optimal design problems.
Supremal functionals

$$\begin{cases} \text{Dimensional reduction} \\ L^p \text{ approximation} \\ \text{Minimum problems without } \textit{convexity}. \end{cases}$$

The results have been obtained in collaboration with
Graça Carita (University of Evora),
Ana Margarida Ribeiro (New University of Lisbon).

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Common feature of the integral problems consists of minimizing a functional
$\mathcal{F}(\cdot,\cdot)$ with dependence on two unrelated fields.

1. $\mathcal{F}(u,v) = \|Du\|(\Omega) + \frac{1}{2\lambda}\|\varphi - u - v\|^2_{L^2}$: φ: given image, u, v: cartoon and noise.

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Examples:

1. $\mathcal{F}(u,v) = \|Du\|(\Omega) + \frac{1}{2\lambda}\|\varphi - u - v\|^2_{L^2}$: φ: given image, u, v: cartoon and noise.
2. $\mathcal{F}(u,v) = \int_\Omega f\left(\frac{x}{\varepsilon}, v, \nabla u\right) dx$ u, v: displacement and chemical composition.

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2. $\mathcal{F}(u,v) = \int_\Omega f\left(\frac{x}{\varepsilon}, v, \nabla u\right) dx$ u, v: displacement and chemical composition.
3. $\mathcal{F}(\chi,v) = \int_\Omega (\chi W_1(\nabla v) + (1-\chi)W_2(\nabla v))dx + |D\chi|(\Omega)$. Ω: design region, v: displacement, χ: set to fill.

Focus on 3: Optimal design problems

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Optimal Design

Main contributions (not complete list)

- Kohn- Strang
- Ambrosio -Buttazzo - with perimeter penalization
- Kohn- Li - with perimeter penalization
- Bouchitté- Fragalá-Seppecher - thin structures and linear elasticity setting)
- Fonseca- Franfort - dimensional reduction with no perimeter penalization
- Braides-Fonseca-Francfort- dimensional reduction with no perimeter penalization
- Larsen
- Carozza-Fonseca-Passarelli di Napoli -regularity

Optimal Design Problem

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3. Let $F : BV(\Omega; \{0,1\}) \times W^{1,p}(\Omega; \mathbb{R}^d) \to [0,+\infty], \ p \geq 1$

$$F(\chi_E, v) \ : \ = \int_\Omega (\chi_E W_1 + (1-\chi_E) W_2)(\nabla v) dx \qquad (1)$$
$$- \int_\Omega f \cdot v \ dx + \mathrm{Per}(\mathrm{E}; \Omega) \ \Big)$$

$$\lambda := \frac{1}{|\Omega|} \int_\Omega \chi_E(x)\, dx \rightsquigarrow \text{ vol. frac.}$$

$E \subset \Omega$ has finite perimeter, $f \in L^{p'}(\Omega)$.
$W_i : \mathbb{R}^{dxN} \to \mathbb{R}$ continuous satisfying

$$\alpha |\xi|^p - \gamma \leq W_i(\xi) \leq \beta(1 + |\xi|^p), \qquad \forall\, \xi \in \mathbb{R}^{d \times N}, \ i = 1, 2, \qquad (2)$$

for some $\alpha, \ \beta, \gamma > 0$.

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$$\mathcal{C}(\chi_E) = -\inf_{v \in W^{1,p}} \left\{ \int_\Omega (\chi_E W_1 + (1-\chi_E) W_2)(\nabla v) dx - \int_\Omega f \cdot v \; dx + \operatorname{Per}(E;\Omega) \right\}$$

The best optimal design would be

$$-\sup_{\chi} \left\{ -\mathcal{C}(\chi) : \chi \in BV(\Omega; \{0,1\}), \; \frac{1}{|\Omega|} \int_\Omega \chi \; dx = \lambda \right\}$$

Problem

$$\inf_{(\chi,v)} \left\{ F(\chi, v) : v = 0 \text{ on } \partial\Omega, \; \frac{1}{|\Omega|} \int_\Omega \chi \; dx = \lambda, \; v \in W^{1,p}(\Omega; R^d), \; \chi \in BV(\Omega; \{0,1\}) \right\}$$

Result: superlinear case

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Theorem ($p > 1$)

$\Omega \subset \mathbb{R}^N$ bounded, Lipschitz, open and $W_i : R^{d\times N} \to [0, +\infty)$, $i = 1, 2$, continuous with growth (2). If $\mathcal{F}$ denotes the relaxation ($L^1 \times L^1$) of F in (1) then

$$\mathcal{F}(\chi, v) = J_p(\chi, v)$$

for every $(\chi, v) \in BV(\Omega; \{0, 1\}) \times W^{1,p}(\Omega; \mathbb{R}^d)$.

$J_p : BV(\Omega; \{0, 1\}) \times W^{1,p}(\Omega; \mathbb{R}^d) \to [0, +\infty[$ is defined as

$$J_p(\chi, v) := \int_\Omega QV(\chi, \nabla v)dx - \int_\Omega f \cdot v dx + |D\chi|(\Omega).$$

Comments

- $$V(e,\xi) := eW_1(\xi) + (1-e)W_2(\xi) \tag{3}$$
- QV stands for the quasiconvexification of V in the second variable.
- The proof is direct via double inequality. It just requires p-equi-integrability results for the upper bound.
- The result is available also in the $3D - 2D$ setting.
- Other penalizations terms can be taken into account.

Comments

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$p > 1$: no interaction!

Linear case for optimal design

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Case $p = 1$: (2) reads as

$$\alpha|\xi| - \gamma \leq W_i(\xi) \leq \beta(|\xi| + 1) \qquad i = 1, 2.$$

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$$\alpha|\xi| - \gamma \leq W_i(\xi) \leq \beta(|\xi| + 1) \quad i = 1, 2.$$

- Relaxation $\mathcal{F}$ of F in (1) $\rightsquigarrow$ requires a representation on $BV(\Omega; \{0, 1\}) \times BV(\Omega; \mathbb{R}^d)$.
- Question: Which interaction between χ and v?

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- Relaxation $\mathcal{F}$ of F in (1) $\rightsquigarrow$ requires a representation on $BV(\Omega; \{0, 1\}) \times BV(\Omega; \mathbb{R}^d)$.
- Question: Which interaction between χ and v?
- Answer: Just in the 'jump set'.

Integral representation for $\mathcal{F}$, $p = 1$

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Theorem

$\Omega \subset \mathbb{R}^N$, $W_i : R^{d\times N} \to [0, +\infty)$, $i = 1, 2$, *continuous with linear growth. Then, for every* $(\chi, v) \in BV(\Omega; \{0, 1\}) \times BV(\Omega; \mathbb{R}^d)$,

$$\mathcal{F}(\chi, v) = J_1(\chi, v),$$

where

$$J_1(\chi, v) = \int_\Omega QV(\chi, \nabla v)dx + \int_\Omega QV^\infty \left(\chi, \frac{dD^c v}{d\,|D^c v|}\right) d\,|D^c v|$$

$$+ \int_{J_{(\chi,v)}} K_2\left(\chi^+, \chi^-, v^+, v^-, \nu\right) d\mathcal{H}^{N-1}.$$

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Recall that V is as in (3) and QV is its quasiconvexification

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Recall that V is as in (3) and QV is its quasiconvexification and QV^{∞} is the recession function of QV, namely,

$$QV^{\infty}(e,\xi) := \lim_{t\to\infty} \frac{QV(e, t\xi)}{t},$$

Relaxation: integral representation for $\mathcal{F}$, $p=1$.

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The interaction is described through the following density

$$K_2(a,b,c,d,\nu) := \inf\left\{\int_{Q_\nu} QV^\infty(\chi(x),\nabla v(x))dx + |D\chi|(Q_\nu) : (\chi,v)\in\mathcal{A}_2(a,b,c,d,\nu)\right\}, \tag{4}$$

where

$$\mathcal{A}_2(a,b,c,d,\nu) := \Big\{(\chi,v)\in BV(Q_\nu;\{0,1\})\times W^{1,1}\left(Q_\nu;\mathbb{R}^d\right) : (\chi,v)=(a,c) \text{ if } y\cdot\nu=\frac{1}{2},\ (\chi,v)=(b,d) \text{ if } y\cdot\nu=-\frac{1}{2}, (\chi,v)1-\text{periodic in } \nu_1,\dots,\nu_{N-1} directions\Big\},$$

for $(a,b,c,d,\nu)\in\{0,1\}\times\{0,1\}\times\mathbb{R}^d\times\mathbb{R}^d\times S^{N-1}$, with Q_ν the unit cube, centered at the origin, with axes parallel to $\{\nu_1,\nu_2,\dots,\nu_{N-1},\nu\}$.

Relaxation: representation for $\mathcal{F}$, $p = 1$.

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- The surface energy can be specialized as follows

$$\int_{J_{(\chi,v)}} K_2(\chi^+,\chi^-,v^+,v^-,\nu_{(\chi,v)})d\mathcal{H}^{N-1} =$$

$$\int_{J_v \setminus J_\chi} QV^\infty(\chi,(v^+ - v^-)\otimes \nu_v)d\mathcal{H}^{N-1} +$$

$$|D\chi|(\Omega \cap (J_\chi \setminus J_v))$$

$$+ \int_{J_\chi \cap J_v} K_2(\chi^+,\chi^-,v^+,v^-,\nu_{(\chi,v)})d\mathcal{H}^{N-1}.$$

Representation for $\mathcal{F}$, $p = 1$: comments.

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- Key idea: to consider the couple (χ, v) as a unique BV-field U, keeping in mind that $\nabla U = (0, \nabla v)$ and $D^c U = (0, D^c v)$

Representation for $\mathcal{F}$, $p = 1$: comments.

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- Key idea: to consider the couple (χ, v) as a unique BV-field U, keeping in mind that $\nabla U = (0, \nabla v)$ and $D^c U = (0, D^c v)$
- Existing results: Barroso-Bouchitté-Buttazzo-Fonseca ARMA (1996): Relaxation in $BV*$ of $u \in SBV \to \int_\Omega f(x, \nabla u)dx + \int_{J_u} g(x, [u](x), \nu_u)d\mathcal{H}^{N-1}$. Bouchitté -Fonseca- Mascarenhas, Global Method ARMA (1998).

Representation for $\mathcal{F}$, $p = 1$. Sketch of the proof.

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- Blow-up!

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- Blow-up!
- Lower bound inequality:

$$\mathcal{F}(\chi, v) \geq \underline{\lim}_{\varepsilon \to 0} \int_\Omega (\chi_\varepsilon W_1(\nabla v_\varepsilon) + (1 - \chi_\varepsilon) W_2(\nabla v_\varepsilon))\, dx. \tag{5}$$

"Bulk" and "Cantor" parts: it 's enough.

Representation for $\mathcal{F}$, $p = 1$. Sketch of the proof.

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- Blow-up!
- Lower bound inequality:

$$\mathcal{F}(\chi, v) \geq \lim_{\varepsilon \to 0} \int_\Omega (\chi_\varepsilon W_1(\nabla v_\varepsilon) + (1 - \chi_\varepsilon) W_2(\nabla v_\varepsilon))\, dx. \tag{5}$$

  "Bulk" and "Cantor" parts: it 's enough.
- But
  $\mathcal{F}(\chi, v) > \lim_{\varepsilon \to 0} \int_\Omega (\chi_\varepsilon W_1(\nabla v_\varepsilon) + (1 - \chi_\varepsilon) W_2(\nabla v_\varepsilon))\, dx.$
  need of *'extending-gluing'* Fonseca-Muller ARMA (1993) and Ambrosio-Braides JMPA (1990).

Representation for $\mathcal{F}$, $p = 1$. Sketch of the proof.

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- Upper bound inequality: $\mathcal{F}$ is a measure absolutely continuous with respect to $\mathcal{L}^N + |D\chi| + |Dv|$.

Representation for $\mathcal{F}$, $p = 1$. Sketch of the proof.

- Upper bound inequality: $\mathcal{F}$ is a measure absolutely continuous with respect to $\mathcal{L}^N + |D\chi| + |Dv|$.
- "Bulk" and "Cantor" parts :via "Global Method for Relaxation" BFM (1998).

Representation for $\mathcal{F}$, $p = 1$. Sketch of the proof.

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- Upper bound inequality: $\mathcal{F}$ is a measure absolutely continuous with respect to $\mathcal{L}^N + |D\chi| + |Dv|$.
- "Bulk" and "Cantor" parts :via "Global Method for Relaxation" BFM (1998).
- "Jump" term requires a direct estimate.

Representation for $\mathcal{F}$: open points.

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- Consider

$$\inf \left\{ \underline{\lim}_{\varepsilon \to 0} \left(\int_\Omega (u_\varepsilon W_1 + (1-u_\varepsilon) W_2)(\nabla v_\epsilon)\, dx + \int_\Omega |\nabla v_\varepsilon| dx \right) : \{(u_\varepsilon, v_\varepsilon)\} \subset W^{1,1}(\Omega;[0,1]) \times W^{1,1}(\Omega;\mathbb{R}^d) \right.$$

$$\left. \begin{array}{l} u_\varepsilon \to u \text{ in } L^1(\Omega;[0,1]) \\ v_\varepsilon \to v \text{ in } L^1\left(\Omega;\mathbb{R}^d\right), \end{array} \right\},$$

Representation for $\mathcal{F}$: open points.

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- Consider

$$\inf\left\{\varliminf_{\varepsilon\to 0}\left(\int_\Omega (u_\varepsilon W_1+(1-u_\varepsilon)W_2)(\nabla v_\epsilon)\,dx+\int_\Omega |\nabla v_\varepsilon|dx\right) : \{(u_\varepsilon,v_\varepsilon)\}\subset W^{1,1}(\Omega;[0,1])\times W^{1,1}(\Omega;\mathbb{R}^d)\quad \begin{array}{l} u_\varepsilon\to u \text{ in } L^1(\Omega;[0,1]) \\ v_\varepsilon\to v \text{ in } L^1\left(\Omega;\mathbb{R}^d\right), \end{array}\right\},$$

- Particular case in Fonseca-Muller ARMA (1993).

Representation for $\mathcal{F}$: open points.

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- Recall that the jump term in FM (1993) is given by :

$$K(a,b,c,d,\nu) := \inf\left\{\int_{Q_\nu} (QV^\infty(u,\nabla v) + |\nabla u|)\,dx : (u,v) \in \mathcal{A}(a,b,c,d,\nu)\right\},$$

where V is as in (3) and

$$\mathcal{A}(a,b,c,d,\nu) := \left\{(u,v) \in W^{1,1}\left(Q_\nu;\mathbb{R}^{1+d}\right) : (u,v) = (a,c) \text{ if } y\cdot\nu = \tfrac{1}{2},\ (u,v) = (b,d) \text{ if } y\cdot\nu = -\tfrac{1}{2}, (u,v) \text{ are } 1-\text{periodic in } \nu_1,\ldots,\nu_{N-1} directions\right\},$$

$$(a,b,c,d,\nu) \in \{0,1\} \times \{0,1\} \times \mathbb{R}^d \times \mathbb{R}^d \times S^{N-1}.$$

Representation for $\mathcal{F}$: open points.

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where V is as in (3) and

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$$(a,b,c,d,\nu) \in \{0,1\}\times\{0,1\}\times\mathbb{R}^d\times\mathbb{R}^d\times S^{N-1}.$$

- Easy to see that $K \le K_2$.What about $K_2 \le K$?

Optimal design: generalizations

- Other growth assumptions on W_1 and W_2.

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- Other growth assumptions on W_1 and W_2.
- Replace perimeters by piecewise affine functions.

Optimal design: generalizations

- Other growth assumptions on W_1 and W_2.
- Replace perimeters by piecewise affine functions.
- Relaxation in $SBV_0\left(\Omega;\mathbb{R}^m\right)\times W^{1,1}\left(\Omega;\mathbb{R}^d\right)$

$$\mathcal{F}\left(u,v\right)=$$

$$\inf\left\{\varliminf_{\varepsilon\to 0}\left(\int_\Omega f\left(u_\varepsilon,\nabla v_\varepsilon\right)dx+\int_{J_{u_\varepsilon}} g\left(u_\varepsilon^+,u_\varepsilon^-,\nu_{u_\varepsilon}\right)d\mathcal{H}^{N-1}\right)\right.$$

$$\left\{\left(u_\varepsilon,v_\varepsilon\right)\right\}\subset SBV_0\left(\Omega;\mathbb{R}^m\right)\times W^{1,1}\left(\Omega;\mathbb{R}^d\right),$$

$$\left.\begin{array}{l} u_\varepsilon\to u \text{ in } L^1\left(\Omega;\mathbb{R}^m\right)\\ v_\varepsilon\to v \text{ in } L^1\left(\Omega;\mathbb{R}^d\right)\end{array}\right\}$$

Optimal design: generalizations.

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Our result says that

$$\mathcal{F}(u,v) = \int_\Omega Qf\left(u, \nabla v\right) dx + \int_\Omega Qf^\infty \left(u, \frac{dD^c v}{d\left|D^c v\right|}\right) d\left|D^c v\right| +$$

$$\int_{J_{(u,v)}\cap A} K_3\left(u^+, u^-, v^+, v^-, \nu\right) d\mathcal{H}^{N-1},$$

for every $(u,v) \in SBV_0(\Omega;\mathbb{R}^m) \times BV(\Omega;\mathbb{R}^d)$.

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$$K_3(a,b,c,d,\nu) :=$$

$$\inf\left\{\int_{Q_\nu} Qf^\infty(u,\nabla v)dx + \int_{J_u\cap Q_\nu} g(u^+,u^-,\nu)d\mathcal{H}^{N-1} :\right.$$

$$\left.(u,v)\in\mathcal{A}_3(a,b,c,d,\nu)\right\}$$

where

$$\mathcal{A}_3\left(a,b,c,d,\nu\right) := \Big\{(u,v)\in(SBV_0(Q_\nu;\mathbb{R}^m)\cap L^\infty(Q_\nu;\mathbb{R}^m))\times W^{1,1}(Q_\nu;\mathbb{R}^d) : (u,v)=(a,c) \text{ if } y\cdot\nu=-\frac{1}{2}, (u,v)=(b,d) \text{ if } y\cdot\nu=\frac{1}{2}, (u,v) \text{ are } 1-\text{periodic in } \nu_1,\dots\nu_{N-1}\text{directions}\Big\}, \tag{6}$$

Optimal design: generalizations.

- Comparison: $K_3 \leq K_2$. We suspect $K_3 < K_2$!

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Optimal design: generalizations.

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- Comparison: $K_3 \leq K_2$. We suspect $K_3 < K_2$!
- More explicit formulas for the surface term K_3.

$$\int_{J_{(u,v)}} K_3(u^+, u^-, v^+, v^-, \nu_{(u,v)})d\mathcal{H}^{N-1} =$$

$$\int_{J_v \setminus J_v} Qf^\infty(u, (v^+ - v^-) \otimes \nu_v)d\mathcal{H}^{N-1} +$$

$$\int_{J_u \setminus J_v} \mathcal{R}g(u^+, u^-, \nu_u)d\mathcal{H}^{N-1} +$$

$$\int_{J_u \cap J_v} K_3(u^+, u^-, v^+, v^-, \nu_{(u,v)})d\mathcal{H}^{N-1}$$

$\mathcal{R}g$ is the BV-elliptic envelope of g!

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Aronsson in

- *Minimization Problems for the Functional* $\sup_x F(x, f(x), f'(x))$. Ark. Mat. **6** (1965), 33- 53,
- *Min. Pb....* Ark. Mat. **6** (1966), 409- 431,
- *Min. Pb...* Ark. Mat.**7** (1969), 509- 512, and in
- Extension of Functions satisfying Lipschitz conditions. Ark. Mat. **6** (1967), 551- 561

developed the theory in connection with infinity laplacian namely with

$$\Delta_\infty u = 0 \text{ in } \Omega \subset \mathbb{R}^N, \quad \text{equivalently } <\nabla u, D^2 u \nabla u>= 0,$$

$$(i.e. u_{x_i} u_{x_j} u_{x_i x_j} = 0,)$$

with $u : \Omega \subset \mathbb{R}^N \to \mathbb{R}$, u smooth enough

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Namely he considered the L^∞-variational problem

$$\min I_\infty(u) = \min \operatorname*{ess\,sup}_{\Omega} |\nabla u|.$$

for which

$$\Delta_\infty u = 0$$

represents the Euler-Lagrange (now 'Euler-Aronsson') equation.

Barron et al. (20-25 years later) addressed the theory, i.e.

$$\min_{u \in W^{1,\infty}(\Omega)} \operatorname*{ess\,sup}_{x \in \Omega} f(x, u(x), \nabla u(x))$$

starting the study of $\operatorname{ess\,sup}_{x\in\Omega} f(x, u(x), \nabla u(x))$ making a parallel with the integral setting.

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Why?

- Newton's law in $\mathbb{R}$. Pb: Want to minimize the maximum pointwise force of an object moving :$\min_u \|\nu(x)u''(x)\|_{L^\infty}$, $\nu(x)$ is the proportional factor between force and acceleration and it may be media dependent and discontinuous...
- Chebychev approximation theory,
- Design of beam subjected to loading and bending needs to minimize L^∞ norm of second derivatives of deflection.
- Minimal Lipschitz extension.

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- Chebychev approximation theory,
- Design of beam subjected to loading and bending needs to minimize L^∞ norm of second derivatives of deflection.
- Minimal Lipschitz extension.

Take $u_0 \in W^{1,\infty}(\Omega)$ ($f = |\ |$) and consider

$$\inf \left\{ \operatorname*{ess\,sup}_{x\in\Omega} f(\nabla u(x)) : u \in u_0 + W_0^{1,\infty}(\Omega) \right\}, \qquad \text{(P)}$$

Supremal functionals - main contributions:

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- Lower semicontinuity: by Barron, Jensen, Prinari,...
- Control theory: Barron,...
- PDE's applications: Barron,...
- L^p-approximation/ Γ-convergence: De Pascale, Prinari, Garroni, Ponsiglione,..
- Applications: Nesi, Bocea, Garroni, Prinari,...
- Supremal representation: Barron, Cardaliaguet, Prinari,...
- Homogenization: Briani, Garroni, Prinari,...
- Dimension reduction: Prinari, Babadjian, Z.
- Absolute minimizers: Champion, De Pascale, Crandall, Jutineen, Jensen, Barron,...
- $\mathcal{A}_\infty$-quasiconvexity: Bocea, Ansini, Prinari,...
- BV-setting: Gori, De Cicco, Maggi, Braides,...

Supremal functionals: basic facts

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Focus on $F(u) = \operatorname{ess\,sup}_{x \in \Omega} f(\nabla u(x)), \quad u \in W^{1,\infty}(\Omega).$

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Barron and Liu in Appl. Math. Optim. '97 rediscovered the notion of 'level convexity', in order to characterize lower semicontinuity for F (cf. also Acerbi-Buttazzo-Prinari)

Theorem - Necessity If $F(u)$ is sequentially weak* lower semicontinuous, then f is lower semicontinuous and level convex.

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Theorem - Sufficiency (Barron-Jensen-Wang '99 Ann. Poinc.) If $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ level convex and lower semicontinuous then $F(u)$ is sequentially weakly* lower semicontinuous.

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Level convexity

Definition $f : \mathbb{R}^N \to \mathbb{R}$ is 'level convex' if the sublevels are convex, i.e. $\forall t \in \mathbb{R}$

$$L_t(f) := \{\xi \in \mathbb{R}^N : f(\xi) \leq t\} \text{ is convex.}$$

In other words for every $\lambda \in [0,1], \xi_1, \xi_2 \in \mathbb{R}^N$

$$f(\lambda\xi_1 + (1-\lambda)\xi_2) \leq \max\{f(\xi_1), f(\xi_2)\}$$

or **Supremal Jensen's ineq.**

$$f\left(\int_\Omega v(x)d\mu(x)\right) \leq \mu - \operatorname{ess}\sup_{x\in\Omega}(f(v(x))),$$

with μ probability measure, $v \in L^1_\mu(\Omega)$, Ω bounded domain.

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Level convexity

Convexity implies *level convexity*. Consider $f : x \in (0,1) \to \sqrt{x}$ to see that *level convexity* does not imply convexity.

Level convexity

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Level convexity does not imply lower semicontinuity, i.e. level sets can be convex but not closed. Take in $\mathbb{R}$, $f := \chi_{\mathbb{R}\setminus(0,1)}$.

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Lower semicontinuous envelopes of level convex functions $f : \mathbb{R}^n \to [-\infty, +\infty]$ are level convex.

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Lower semicontinuous envelopes of level convex functions $f : \mathbb{R}^n \to [-\infty, +\infty]$ are level convex.

Level convex function $f : \mathbb{R}^n$, $n > 1$, may not be Borel measurable: i.e. $f := \chi_{\mathbb{R}^n\setminus C}$, C convex but not Borel measurable.

Relaxation Theorem

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Theorem

*$\Omega \subset \mathbb{R}^n$ bn'd open set, Lipschitz bd. and $f : \mathbb{R}^n \to \mathbb{R}$ continuous (**and coercive**). If $F(u) := \text{ess sup}_{x\in\Omega} f(\nabla u(x))$ for $u \in u_0 + W_0^{1,\infty}(\Omega)$, and $\overline{F}$ is its relaxed functional, then*

$$\overline{F}(u) = \operatorname*{ess\,sup}_{x\in\Omega} f^{\mathrm{lslc}}(\nabla u(x)), \ \forall\ u \in u_0 + W_0^{1,\infty}(\Omega) \tag{7}$$

where f^{lslc} is the lower semicontinuous and level convex envelope of f,

$$f^{\mathrm{lslc}}(\xi) = \sup\{g(\xi) : g \leq f, g \text{ lsc and level convex}\}. \tag{8}$$

Representation for f^{lslc}

Many 'equivalent' representations are available for f^{lslc}.

Caratheodory's formula (f lsc and $\displaystyle\lim_{|\xi|\to+\infty}\frac{f(\xi)}{|\xi|}=+\infty$):

$$f^{\mathrm{lslc}}(\xi) := \min\left\{\max_{1\le i\le n+1} f(\xi_i) : \xi=\sum_{i=1}^{n+1}\lambda_i\xi_i, \lambda_i\ge 0, \sum_{i=1}^{n+1}\lambda_i=1\right\}.$$

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In particular if f is continuous and $\displaystyle\lim_{|\xi|\to+\infty} f(\xi) = +\infty$,

$$f^{\mathrm{lslc}}(\xi) = \sup\{g(\xi), g \text{ level convex }, g \leq f\} =: f^{\mathrm{lc}}(\xi).$$

In terms of duality: define $f^* : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, and $f^{**} : \mathbb{R}^n \to \mathbb{R}$ as

$$f^*(p, \gamma) := \sup\{\min\{< p, x >, \gamma\} - f(x) : x \in \mathbb{R}^n\},$$

$$f^{**}(x) := \sup\{\min\{< p, x >, \gamma\} - f^*(p, \gamma) : p \in \mathbb{R}^n, \gamma \in \mathbb{R}\}.$$

Representation for f^{lslc}

Theorem

If $f : \mathbb{R}^n \to [-\infty, \infty]$ is bounded from below, then

$$f^{**}(\xi) = f^{\mathrm{lslc}}(\xi) \text{ for every } \xi \in \mathbb{R}^n. \tag{9}$$

Ideas about proof: Ekeland & Temam, i.e. Separation theorems (c.f. Martinez-Legaz '88 and Barron & Liu '97).

Representation for f^{lslc}

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I. E.: f level convex and lsc. If $(\xi_0, b) \notin \mathrm{epi}(f)$, then $\exists p \in \mathbb{R}^n$, $\gamma, \beta \in \mathbb{R}$ such that

$$\min\{< p, \xi_0 >, \gamma\} - \beta = b,$$

$$\min\{< p, \xi >, \gamma\} - \beta < \eta, \quad \forall(\xi, \eta) \in \mathrm{epi}(f).$$

Representation for f^{lslc}

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If $f : \mathbb{R}^n \to [-\infty, \infty]$ *is bounded from below, then*

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$$\min\{< p, \xi >, \gamma\} - \beta < \eta, \quad \forall (\xi, \eta) \in \mathrm{epi}(f).$$

Moreover

$$f^{**}(\xi) = \sup\{g(\xi) : g \text{ is level affine and } g(\xi) \leq f(\xi)\} = f^{\mathrm{lslc}}(\xi).$$

Relaxation Theorem - equivalent formulation:

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Take $F : u \in u_0 + W_0^{1,\infty}(\Omega) \to \operatorname{ess\,sup}_{x\in\Omega} f(\nabla u(x))$,

$\overline{F}$ be its relaxed functional,

$F^{\mathrm{lslc}} = \sup\{G, G \le F, \text{weakly} * \text{lsc, level convex}\}$,

$F^* : (L^1(\Omega))^{n+1} \times \mathbb{R} \to [-\infty, \infty]$, defined by

$F^*(p,\gamma) = \sup\{\min\{< p, u >, \gamma\} - F(u), u \in u_0 + W_0^{1,\infty}(\Omega)\}$;

$F^{**} : u_0 + W_0^{1,\infty}(\Omega) \to [-\infty, \infty]$, defined by $F^{**}(u) = \sup\{\min\{< p, u >, \gamma\} - F^*(p,\gamma) : p \in (L^1(\Omega))^{n+1}, \gamma \in \mathbb{R}\}$

$< \cdot, \cdot >$ is the duality between L^1 and L^∞, where $u \in W^{1,\infty}(\Omega)$ is identified with $(u, \nabla u) \in (L^\infty(\Omega))^{n+1}$.

Relaxation Theorem - equivalent formulation:

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$< \cdot, \cdot >$ is the duality between L^1 and L^∞, where $u \in W^{1,\infty}(\Omega)$ is identified with $(u, \nabla u) \in (L^\infty(\Omega))^{n+1}$.

Theorem

$$F^{\text{lslc}}(u) = \overline{F}(u) = F^{**}(u) = \operatorname*{ess\,sup}_{x \in \Omega} f^{**}(\nabla u(x)),$$

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Want to find solutions to (P). So consider again

$$\inf \left\{ \operatorname*{ess\,sup}_{x \in \Omega} f(\nabla u(x)) : u \in u_0 + W_0^{1,\infty}(\Omega) \right\}, \qquad \text{(P)}$$

$$\inf \left\{ \operatorname*{ess\,sup}_{x \in \Omega} f^{\mathrm{lslc}}(\nabla u(x)) : u \in u_0 + W_0^{1,\infty}(\Omega) \right\}. \qquad (P^{lc})$$

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$$\inf\left\{\operatorname*{ess\,sup}_{x\in\Omega} f^{\mathrm{lslc}}(\nabla u(x)) : u \in u_0 + W_0^{1,\infty}(\Omega)\right\}. \qquad (P^{lc})$$

Integral setting: A. Cellina Nonlinear Analysis (1993)

$$\inf\left\{\int_\Omega f(\nabla u(x))dx : u \in u_0 + W_0^{1,\infty}(\Omega)\right\} \qquad (P')$$

$$\inf\left\{\int_\Omega Cf(\nabla u(x))dx : u \in u_0 + W_0^{1,\infty}(\Omega)\right\}. \qquad (P'^{c})$$

$(P) = (P^{\mathrm{lc}})$

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Theorem

Let $\Omega \subset \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ be as before. Let $u_0 \in W^{1,\infty}(\Omega)$ and let (P) and (P^{lc}) be as above. Then

$$(P) = (P^{\mathrm{lc}}).$$

Moreover, if the boundary condition is affine, say $u_0(x) = u_{\xi_0}(x) := \xi_0 \cdot x + c$ with $\xi_0 \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$, then

$$(P) = f^{\mathrm{lslc}}(\xi_0).$$

Proof: $(P) = (P^{\mathrm{lc}})$ follows by relaxation.
((P^{lc}) is a minimum, i.e. $\exists\, \overline{u} \in u_0 + W_0^{1,\infty}(\Omega)$ such that

$$\overline{F}(\overline{u}) = (P^{\mathrm{lc}}).$$

The case of affine boundary conditions, is supremal Jensen inequality.

Another formula for f^{lslc}

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By-product-in the scalar case: **when f is continuous and coercive**, for every $\xi_0 \in \mathbb{R}^n$,

$$f^{\mathrm{lslc}}(\xi_0) = \inf \left\{ \operatorname*{ess\,sup}_{x\in\Omega} f(\nabla u) : \ u \in u_{\xi_0} + W_0^{1,\infty}(\Omega) \right\} \quad (10)$$

(cf. Barron-Jensen-Wang '99), Weak Morrey-quasiconvexity: $f(\xi_0) = \inf \left\{ \mathrm{ess\,sup}_{x\in Q} f(\nabla u(x)) : \ u \in u_{\xi_0} + W_0^{1,\infty}(Q) \right\}$- necessary condition for lower semicontinuity in the vectorial case.

Another formula for f^{lslc}

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(cf. Barron-Jensen-Wang '99), Weak Morrey-quasiconvexity: $f(\xi_0) = \inf\left\{ \operatorname{ess\,sup}_{x\in Q} f(\nabla u(x)) : \; u \in u_{\xi_0} + W_0^{1,\infty}(Q) \right\}$- necessary condition for lower semicontinuity in the vectorial case.

[Proof]: Enough: $\forall \epsilon > 0$ find $u \in u_{\xi_0} + W_0^{1,\infty}(\Omega)$ such that

$$f^{\mathrm{lslc}}(\xi_0) + \epsilon \geq \operatorname*{ess\,sup}_{x\in\Omega} f(\nabla u(x)),$$

equiv. find $u \in u_{\xi_0} + W_0^{1,\infty}(\Omega)$ s.t. $\nabla u(x) \in L_{f^{\mathrm{lslc}}(\xi_0)+\epsilon}(f)$ a.e. $x \in \Omega$.

Thus we are led to solve a differential inclusion.

Existence of solutions for the non-level convex problem (P).

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Theorem (NSC)

Consider (P) and (P^{lc}) as above, (i.e. $\Omega, f : \mathbb{R}^n \to \mathbb{R}$ and $u_0 \in W^{1,\infty}(\Omega)$).

(P) has a solution if and only if there exists $u \in u_0 + W_0^{1,\infty}(\Omega)$ such that

$$f(\nabla u(x)) \leq (P^{\mathrm{lc}}), \ \ a.e.\ x \in \Omega. \tag{11}$$

In particular, if u_0 is affine, i. e. $u_0 = u_{\xi_0}$ then condition (11) becomes

$$f(\nabla u(x)) \leq f^{\mathrm{lslc}}(\xi_0), \ \ a.e.\ x \in \Omega. \tag{12}$$

Moreover, still if $u_0 = u_{\xi_0}$, problem (P) admits a solution if and only if

$$\xi_0 \in L_{f^{\mathrm{lslc}}(\xi_0)}(f) \cup \mathrm{int}\, L_{f^{\mathrm{lslc}}(\xi_0)}(f^{\mathrm{lslc}}). \tag{13}$$

Connections with the integral setting (cf. (11) and (12))

Comparing with the analogous problem in the integral setting (scalar case, f not convex),

$$\inf \left\{ \int_\Omega f(\nabla u(x))dx : u \in u_0 + W_0^{1,\infty}(\Omega) \right\},$$

we recall that the correspondent formulas to (11) and (12) are given by

$$f(\nabla \overline{u}(x)) = Cf(\nabla \overline{u}(x)) \text{ a. e. in } \Omega, \tag{14}$$

and

$$\int_\Omega Cf(\nabla \overline{u}(x))dx = Cf(\xi_0)\mathcal{L}(\Omega), \tag{15}$$

where $\overline{u} \in u_0 + W^{1,\infty}(\Omega)$ and Cf stands for the convex envelope of f and in the affine case recall $u_0 = u_{\xi_0}$.

Tool: Differential Inclusions (cf. Cellina, Dacorogna, Friesecke, Marcellini)

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Theorem (NDI)

Let $\Omega \subset R^n$ *be a bounded open set,* $E \subset \mathbb{R}^n$, $\xi_0 \in \mathbb{R}^n$. *If* $u \in u_{\xi_0} + W_0^{1,\infty}(\Omega)$ *is such that*

$$\nabla u(x) \in E, \ \ a.e.\ x \in \Omega, \textit{then } \xi_0 \in E \cup \operatorname{int} \operatorname{co} E,$$

Tool: Differential Inclusions (cf. Cellina, Dacorogna, Friesecke, Marcellini)

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Theorem (NDI)

Let $\Omega \subset R^n$ be a bounded open set, $E \subset \mathbb{R}^n$, $\xi_0 \in \mathbb{R}^n$. If $u \in u_{\xi_0} + W_0^{1,\infty}(\Omega)$ is such that

$$\nabla u(x) \in E, \text{ a.e. } x \in \Omega, \text{then } \xi_0 \in E \cup \operatorname{int} \operatorname{co} E,$$

Theorem (SDI)

Let $\varphi \in W^{1,\infty}(\Omega)$ satisfying

$$\nabla \varphi(x) \in E \cup \operatorname{int} \operatorname{co} E, \text{ a.e. in } \Omega.$$

Then $\exists u \in \varphi + W_0^{1,\infty}(\Omega)$ s. t. $\nabla u(x) \in E$, a.e. in Ω. Moreover, given $\varepsilon > 0$, u can be chosen such that $||u - \varphi||_{L^\infty(\Omega)} \leq \varepsilon$.

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Back to Theorem NSC

Theorem (NSC)

Let $\Omega \subset \mathbb{R}^n$ *and* $f : \mathbb{R}^n \to \mathbb{R}$ *be as above. Let* $u_0 \in W^{1,\infty}(\Omega)$*. Then problem* (P) *has a solution if and only if there exists* $u \in u_0 + W_0^{1,\infty}(\Omega)$ *such that*

$$f(\nabla u(x)) \leq (P^{\mathrm{lc}}), \; \text{a.e. } x \in \Omega. \quad (11)$$

In particular, if u_0 *is affine, i. e.* $u_0 = u_{\xi_0}$ *then condition (11) becomes*

$$f(\nabla u(x)) \leq f^{\mathrm{lslc}}(\xi_0), \; \text{a.e. } x \in \Omega. \quad (12)$$

Moreover, still if $u_0 = u_{\xi_0}$*, problem (P) admits a solution if and only if*

$$\xi_0 \in L_{f^{\mathrm{lslc}}(\xi_0)}(f) \cup \operatorname{int} L_{f^{\mathrm{lslc}}(\xi_0)}(f^{\mathrm{lslc}}). \quad (13)$$

Back to Theorem NSC-few things

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The last assertion of the Theorem, if u_0 is not affine, says that

$$\nabla u_0(x) \in L_{(P^{\mathrm{lc}})}(f) \cup \operatorname{int} L_{(P^{\mathrm{lc}})}(f^{\mathrm{lslc}}), \text{ a.e. } x \in \Omega$$

is a sufficient condition for existence of solutions to problem (P).

Back to Theorem NSC-few things

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The last assertion of the Theorem, if u_0 is not affine, says that

$$\nabla u_0(x) \in L_{(P^{\rm lc})}(f) \cup \operatorname{int} L_{(P^{\rm lc})}(f^{\rm lslc}), \text{ a.e. } x \in \Omega$$

is a sufficient condition for existence of solutions to problem (P).
If $u \in u_0 + W_0^{1,\infty}(\Omega)$ solves problem (P) then Theorem $[(P) = (P^{\rm lc})]$, implies

$$\operatorname*{ess\,sup}_{x\in\Omega} f\left(\nabla u\left(x\right)\right) = (P^{\rm lc}), \text{ i.e. } (11)$$

For the reverse implication observe that $(P^{\rm lc}) \leq (P)$.

(12) is supremal Jensen's ineq.

(13) follows from Theorems [NDI] and [SDI] (with $E = L_{f^{\rm lslc}(\xi_0)}(f)$) and from $\operatorname{co}(L_{f^{\rm lslc}(\xi_0)}(f)) = L_{f^{\rm lslc}(\xi_0)}(f^{\rm lslc})$.

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A solution always exists in dimension 1

Corollary

Let $\Omega \subset \mathbb{R}$ and $f : \mathbb{R} \to]-\infty; +\infty[$ be as above. Consider (P) with $u_0 = u_{\xi_0}$. Then problem (P) admits a solution for every $\xi_0 \in \mathbb{R}$.

Proof. Want to show that $\xi_0 \in \mathbb{R}$ verifies (13):

$$\xi_0 \in L_{f^{\mathrm{lslc}}(\xi_0)}(f) \cup \operatorname{int} L_{f^{\mathrm{lslc}}(\xi_0)}(f^{\mathrm{lslc}}).$$

Clearly $\xi_0 \in L_{f^{\mathrm{lslc}}(\xi_0)}(f^{\mathrm{lslc}}) = \operatorname{co} L_{f^{\mathrm{lslc}}(\xi_0)}(f)$.
If $\xi_0 \in L_{f^{\mathrm{lslc}}(\xi_0)}(f)$ OK.
W.l.g. $\xi_0 \in \operatorname{co} L_{f^{\mathrm{lslc}}(\xi_0)}(f) \setminus L_{f^{\mathrm{lslc}}(\xi_0)}(f)$.
We conclude that $\xi_0 \in \operatorname{int} L_{f^{\mathrm{lslc}}(\xi_0)}(f^{\mathrm{lslc}}) = \operatorname{int} \operatorname{co} L_{f^{\mathrm{lslc}}(\xi_0)}(f)$: in $\mathbb{R}$, the elements of the boundary of a convex hull belong either to the original set or to the complement of the convex hull.

Counterexample in dimension 2

The set $\operatorname{int} \operatorname{co} L_{f^{\mathrm{lslc}}(\xi_0)}(f)$ might be empty.

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Counterexample in dimension 2

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The set $\operatorname{int}\operatorname{co} L_{f^{\mathrm{lslc}}(\xi_0)}(f)$ might be empty.

Example

Let $f : (z_1, z_2) \in \mathbb{R}^2 \to (z_1^2 - 1)^2 + z_2^2 \in \mathbb{R}$. Clearly $f \geq 0$, and $f(z_1, z_2) = 0$ if and only if $(z_1, z_2) = (\pm 1, 0)$. Since $f \geq 0$ then $f^{\mathrm{lslc}} \geq 0$, and thus

$$\left\{ f^{\mathrm{lslc}} = 0 \right\} = L_0(f^{\mathrm{lslc}}) = \operatorname{co} L_0(f) = [-1, 1] \times \{0\},$$

and it has empty interior.
Take $\xi_0 \in (-1, 1) \times \{0\}$, then
$\nabla u_{\xi_0} \notin L_{f^{\mathrm{lslc}}(\xi_0)}(f) \cup \operatorname{int} L_{f^{\mathrm{lslc}}(\xi_0)}(f^{\mathrm{lslc}})$.
Consider the minimizing problem (P) related to the function f as above, with b.c. $u_0 = u_{\xi_0}$, then (P^{lc}) admits a unique solution.

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Another sufficient condition for existence with more general boundary data.

Theorem

Let $\Omega \subset \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ be as above. Let $u_0 \in W^{1,\infty}(\Omega)$. Consider (P) and (P^{lc}) , and assume that problem (P^{lc}) admits a solution $\overline{u} \in C^1_{piec}(\Omega)$.

If f^{lslc} is constant in each connected component of the set where $f^{\mathrm{lslc}} < f$, then problem (P) has a solution.

Non-existence Results

Theorem (in purpose for f^{lslc})

Let $\Omega \subset \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$ be as above. Consider (P) and (P^{lc}) with the affine boundary condition u_{ξ_0} and assume f^{lslc} satisfies the condition

$$\left.\begin{array}{c} \operatorname*{ess\,sup}_{x\in\Omega} f^{\mathrm{lslc}}(\nabla u(x)) = f^{\mathrm{lslc}}(\xi_0) \\ u \in u_{\xi_0} + W_0^{1,\infty}(\Omega) \end{array}\right\} \Rightarrow \quad u = u_{\xi_0},$$

then problem (P^{lc}) has a unique solution.

So if $f \neq f^{\mathrm{lslc}}$, (P) has no solution!

Non-existence Results

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Theorem (Non Existence)

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $u_{\xi_0}(x) = \xi_0 \cdot x + c$ for some $\xi_0 \in \mathbb{R}^n$ and let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous and strictly level convex function. Then problem

$$\inf \left\{ \operatorname*{ess\,sup}_{x \in \Omega} f\left(\nabla u\left(x\right)\right) : \; u \in u_{\xi_0} + W_0^{1,\infty}(\Omega) \right\}$$

has a unique solution.
Thus (P) with $u_0 = u_{\xi_0}$ for some $\xi_0 \in \mathbb{R}^n \cap \{f^{\mathrm{lslcs}}(\xi) < f(\xi)\}$ has no solution.

Strict level convexity

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A function is said to be strictly level convex if and only if

$$f(t\xi + (1-t)\eta) < \max\{f(\xi), f(\eta)\}$$

for every $t \in (0,1)$ and $\xi \neq \eta$.
Equivalently, in the case f is a Borel measurable and finite function,

$$f\left(\int_\Omega \varphi \, d\mu\right) < \mu - \operatorname*{ess\,sup}_{x \in \Omega} f(\varphi(x)),$$

for every probability measure μ on $\mathbb{R}^n$ supported in the open set $\Omega \subset \mathbb{R}^d$ and every nonconstant $\varphi \in L^1_\mu(\Omega; \mathbb{R}^n)$.

Iff for every c in the range of f, $L_c(f)$ is convex and $R_c(f) \subseteq \mathrm{Ext}(L_c(f))$.

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Crucial observation: In $\mathbb{R}^n, n > 1, f^{lslc}$ is not strictly level convex!
Cf. Caratheodory type formula for f^{lslc}: if $f^{\mathrm{lslc}}(\xi_0) < f(\xi_0)$ and f coercive, then f^{lslc} is constant in a segment line containing ξ_0, eventually with ξ_0 an extreme point.

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Thus nonexistence of solution to (P) **has to be** ensured via weaker conditions on f^{lslc}.

Strict level convexity

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Crucial observation: In $\mathbb{R}^n, n > 1, f^{lslc}$ is not strictly level convex!
Cf. Caratheodory type formula for $f^{\rm lslc}$: if $f^{\rm lslc}(\xi_0) < f(\xi_0)$ and f coercive, then $f^{\rm lslc}$ is constant in a segment line containing ξ_0, eventually with ξ_0 an extreme point.

Thus nonexistence of solution to (P) **has to be** ensured via weaker conditions on f^{lslc}.

Strict level convexity of supremand does not carry to the supremal functional F!

Non existence of solutions

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1. $f : \mathbb{R}^n \to \mathbb{R}$, level convex is *strictly level convex at* $\xi_0 \in \mathbb{R}^n$, if for every $t \in (0,1)$ and for every
$\xi \neq \eta : \xi_0 = t\xi + (1-t)\eta \Rightarrow f(\xi_0) < \max\{f(\xi), f(\eta)\}$,
i.e. $\xi_0 \in \mathrm{Ext} L_{f(\xi_0)}(f)$

2. or for every
$t \in [0,1], \xi \neq \xi_0, f(t\xi_0 + (1-t)\xi) < \max\{f(\xi_0), f(\xi)\}$,

This condition is stronger than **1.**, take

$$f(\xi) = \begin{cases} -\xi & \text{if } \xi \leq 0, \\ 0 & \text{if } \xi > 0. \end{cases}$$

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Other conditions: 3. $f : \mathbb{R}^n \to \mathbb{R}$, level convex is *strictly level convex at* $\xi_0 \in \mathbb{R}^n$ *in at least one direction* if there exists $\alpha \in \mathbb{R}^n \setminus \{0\}$ such that if for some γ and $\eta \in \mathbb{R}^n$,

$$\begin{cases} \xi_0 = t\gamma + (1-t)\eta, t \in (0,1) \\ f(\xi_0) = \max\{f(\gamma), f(\eta)\} \end{cases} \quad \text{then } (\gamma - \eta)\cdot\alpha = 0$$

f strictly level convex at ξ_0 *in at least one direction (and lsc.) iff* $\xi_0 \in \partial L_{f(\xi_0)}(f)$

4.....$\begin{cases} \xi = t\xi_0 + (1-t)\eta, t \in (0,1) \\ f(\xi) = \max\{f(\gamma), f(\eta)\} \end{cases}$ *then* $(\xi_0 - \eta)\cdot\alpha = 0$

$f : \xi \in \mathbb{R}^2 \to \mathrm{dist}(\xi, \mathbb{R}^+ \times \mathbb{R})$ *is satisfying* **3**. *at* $\xi_0 = (0,0)$ *but neither* **1**., **2**. *or* **4**.

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Nonexistence of solutions

Theorem

Let Ω, f be as above. Let $\xi_0 \in \mathbb{R}^n$, consider problem (P) for $u_0 = u_{\xi_0}$ and assume that $f^{\mathrm{lslc}}(\xi_0) < f(\xi_0)$.
(P) has a solution if and only if f^{lslc} is not strictly level convex at ξ_0 in any direction.

[Proof.] It relies on the fact that (P) admits a solution $u \neq u_{\xi_0}$ if $\xi_0 \in \mathrm{int}(L_{f(\xi_0)}(f))$.

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Definition one possible Definition

$u^* \in W^{1,\infty}(\Omega)$ is an absolute minimizer of $F(u,\Omega) := \operatorname*{ess\,sup}_{x\in\Omega} f(\nabla u(x))$, if, for any open set $\Omega' \subset \overline{\Omega'} \subset \Omega$, u^* is a solution for

$$\inf\left\{ \operatorname*{ess\,sup}_{x\in\Omega'} f\left(\nabla u(x)\right) : \; u \in u^* + W_0^{1,\infty}(\Omega') \right\}.$$

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This property is automatically inherited by minimizers of analogous problem in the integral setting.

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It fails for supremals

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Definition one possible Definition
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This property is automatically inherited by minimizers of analogous problem in the integral setting.

It fails for supremals
From now on we will consider Ω convex.

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In the above definition no b.c. have been imposed.

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In the above definition no b.c. have been imposed.

Example

if f level convex, then every affine $u^* = u_\xi$ is an AML!

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In the above definition no b.c. have been imposed.

Example

if f level convex, then every affine $u^* = u_\xi$ is an AML!

(i) f level convex and (P) has u_{ξ_0} as b. c., then u_{ξ_0} is a solution to (P) which is an absolute minimizer.

(ii) If f is strictly level convex at ξ_0 in at least one direction and (P) has u_{ξ_0} as b. c. then u_{ξ_0} is an absolute minimizer and u_{ξ_0} is the only solution to (P). Thus we have uniqueness of the solution to (P). But there is no uniqueness of absolute minimizers since, all the other affine functions are AML.

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Question: Does strict level convexity ensure that minimizers are AML?

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Question: Does strict level convexity ensure that minimizers are AML?

Answer: NO even if b.c. are imposed. I.e. Lipschitz extension problem with b.c. admits more than one solution, but A.M.L. are unique because they solve $\Delta_\infty u = 0$.

AML

Question: Does strict level convexity ensure that minimizers are AML?

Answer: NO even if b.c. are imposed. I.e. Lipschitz extension problem with b.c. admits more than one solution, but A.M.L. are unique because they solve $\Delta_\infty u = 0$.

What about converse? Are AML unique if minimizers unique? Again the answer is no since no b.c. is imposed to define A.M.L. and affine functions answer.

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Observation: We do not know if Crandall's arguments apply to other strict level convex functions (not $\equiv |\cdot|$) to get uniqueness of A.M.L. in terms of PDEs.

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What about AML for non level convex problems?

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What about AML for non level convex problems?
Partial answer: If $f(\xi_0) = f^{\mathrm{lslc}}(\xi_0)$ then ξ_0 is AML for (P).

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Observation: We do not know if Crandall's arguments apply to other strict level convex functions (not $\equiv |\cdot|$) to get uniqueness of A.M.L. in terms of PDEs.

What about AML for non level convex problems?
Partial answer: If $f(\xi_0) = f^{\mathrm{lslc}}(\xi_0)$ then ξ_0 is AML for (P).

In general one should guarantee that $f(\nabla u(x)) \leq (P'^{\mathrm{lc}})$ on any open subset of Ω...

Vectorial case

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There are not relaxation results yet! Why?

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A function f is level convex if and only if all its level sets $L_\alpha(f)$ are convex (for every α), i.e. if and only if $I_{L_\alpha(f)}$ is convex for every α.

$$\left(I_E(x) = \begin{cases} 0 & \text{if } x \in E, \\ +\infty & \text{otherwise.} \end{cases} \right)$$

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A function f is weak Morrey quasiconvex if and only if $I_{L_\alpha(f)}$ is QUASICONVEX in the integral sense! (cf. Barron & Jensen & Wang '01 Ann. I. H. P.)

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But Quasiconvexity in the vectorial integral setting does not have a precise meaning!

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THANK YOU!

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Appendix

STRONG MORREY QUASICONVEXITY: **Definition** A Borel measurable function $f:\mathbb{R}^{m\times n}\to\mathbb{R}$ is said to be strong Morrey quasiconvex if for any $\varepsilon>0$, for any $\xi\in\mathbb{R}^{m\times n}$, and any $K>0$, there exists a $\delta=\delta(\varepsilon,K,\xi)>0$ such that if $\varphi\in W^{1,\infty}(Q;\mathbb{R}^m)$ satisfies

$$\|\nabla\varphi\|_{L^\infty(Q)}\le K, \qquad \max_{x\in\partial Q}|\varphi(x)|\le\delta,$$

then,

$$f(\xi)\le \operatorname*{ess\,sup}_{x\in Q} f(\xi+\nabla\varphi(x))+\varepsilon.$$

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$$\|\nabla\varphi\|_{L^\infty(Q)} \leq K, \qquad \max_{x\in\partial Q} |\varphi(x)| \leq \delta,$$

then,

$$f(\xi) \leq \operatorname*{ess\,sup}_{x\in Q} f(\xi + \nabla\varphi(x)) + \varepsilon.$$

This is the condition proved to be N & S for LSC, but not yet proved to be stronger than Weak M-qcx