Continuity Results and Compactness Tools for Integral Functionals over $BV(\Omega;\mathbb{R}^m)$

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An Extension Problem

Problem (De Giorgi)

What is the ‘right’ way to extend the functional

$$\mathcal{F}[u] := \int_\Omega f(x, u(x), \nabla u(x))\,\mathrm{d}x, \qquad u \in W^{1,1}(\Omega; \mathbb{R}^m),$$

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Motivated by the **Direct Method** for problems with linear growth:

- If $|f(x,y,A)| \geq C|A| - B$, minimising sequences to $\mathcal{F}$ are only compact in $BV(\Omega;\mathbb{R}^m)$, **not** $W^{1,1}(\Omega;\mathbb{R}^m)$.
- For a good existence theory: need to make sense of $\mathcal{F}[u]$ for $u \in BV(\Omega;\mathbb{R}^m)$.

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- Can decompose

$$Du = \overbrace{\nabla u \mathcal{L}^d}^{\text{Classical part}} + \underbrace{D^s u}_{\text{Singular part}}, \quad D^s u = \overbrace{D^j u + D^c u}^{\text{Jump part + Cantor part}}.$$

- $|D^j u| = |u^+ - u^-|\mathcal{H}^{d-1}, \qquad \mathcal{H}^{d-1} << |D^c u| << \mathcal{L}^d$.

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$$u^\theta(x) := \begin{cases} \theta u^+(x) + (1-\theta)u^-(x) & \text{if } x \text{ is a jump point of } u, \\ u(x) & \text{otherwise.} \end{cases}$$

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- If it exists, define the **recession function**

$$f^\infty(x,y,A) := \lim_{\substack{(x',y',A')\to(x,y,A) \\ t\to\infty}} \frac{f(x',y',tA')}{t}.$$

The Scalar Case

-$W^{1,1}(\Omega;\mathbb{R}^m)$ is weak* dense in $BV(\Omega;\mathbb{R}^m)$, so, for $u \in BV(\Omega;\mathbb{R}^m)$, define

$$\mathcal{F}[u] \equiv \mathcal{F}_{**}[u] := \inf \left\{ \liminf_j \mathcal{F}[u_j] : (u_j) \subset W^{1,1}, u_j \overset{*}{\rightharpoonup} u \text{ in } BV \right\}.$$

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-(Dal Maso '80) If $m = 1$ and $f(x, y, \cdot)$ is **convex**, can obtain the **integral representation**

$$\mathcal{F}_{**}[u] = \int_\Omega f\left(x, u, \nabla u\right)\, \mathrm{d}x + \int_\Omega \underbrace{\int_0^1 f^\infty\left(x, u^\theta, \overbrace{\frac{\mathrm{d}D^s u}{\mathrm{d}|D^s u|}}^{\text{Jump direction at } x.}\right) \mathrm{d}\theta}_{\text{Average of } f^\infty \text{ over all values of } u \text{ between jump points.}}\, \mathrm{d}|D^s u|(x).$$

-Also true for $m > 1$ if $f = f(x, A)$ and $f(x, \cdot)$ is **quasiconvex**.

The Vector Valued Case

(Fonseca & Müller '93) If $m > 1$ and $f(x, y, \cdot)$ is quasiconvex,

$$\mathcal{F}_{**}[u] = \int_\Omega f\left(x, u, \nabla u\right)\,\mathrm{d}x + \int_\Omega K\left(x, u^+, u^-, \frac{\mathrm{d}D^j u}{\mathrm{d}|D^j u|}\right)\,\mathrm{d}|D^j u|(x) + \int_\Omega f^\infty\left(x, u, \frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\right)\mathrm{d}|D^c u|(x).$$

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  - Some applications cannot assume this.
  - Relationship with f not clear in this case: in what sense is this an extension?
- Requires other boundedness/continuity assumptions.
- Could another extension of $\int_\Omega f(x, u(x), \nabla u(x))\, \mathrm{d}x$ admit more recovery sequences?

Idea: Is there a topology on $BV(\Omega; \mathbb{R}^m)$ with respect to which we can **continuously** extend $u \mapsto \int_\Omega f(x, u(x), \nabla u(x))\,\mathrm{d}x$?

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Definition

We say that $u_j \to u$ **area-strictly** in $BV(\Omega;\mathbb{R}^m)$ if $u_j \to u$ in $L^1(\Omega;\mathbb{R}^m)$ and

$$\int_\Omega \sqrt{1+|\nabla u_j|^2}\,\mathrm{d}x + |D^s u_j|(\Omega) \to \int_\Omega \sqrt{1+|\nabla u|^2}\,\mathrm{d}x + |D^s u|(\Omega)\,.$$

-Each $u \in BV(\Omega;\mathbb{R}^m)$ can be approximated area-strictly by sequences in $C^\infty(\Omega;\mathbb{R}^m)$ (in particular, by mollifiers).

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Assumptions on f:

Assume that $f\colon \Omega\times\mathbb{R}^m\times M^{m\times d}\to\mathbb{R}$ is Carathéodory and satisfies

- $|f(x,y,A)| \le C(1+|y|^p+|A|)$ for $p\in[1,d/(d-1)]$.
- $f^\infty(x,y,A)$ exists and is continuous $\mathcal{H}^{d-1}$-almost everywhere.

-Optimal hypotheses to ensure that $\mathcal{F}[u]<\infty$.

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Proof strategy:

- Prove for bounded, 1-homogenous f via construction of a special measure in $\mathbf{M}(\Omega \times \mathbb{R}^m; M^{m\times d})$,

$$\mu[u] := \left(\int_0^1 \delta_{u^\theta(x)} \,\mathrm{d}\theta \right) \otimes Du,$$

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For $p = d/d-1$, the approximation argument relies on the following concentration compactness lemma, which is of independent interest:

Lemma (P.L. Lions, '85)

Let $u_j, u \in BV(\Omega; \mathbb{R}^m)$ be such that $u_j \to u$ in L^1, $Du_j \overset{*}{\rightharpoonup} Du$ and $|Du_j|(\Omega) \to |Du|(\Omega)$. Then $u_j \to u$ in $L^{d/d-1}$.

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- **Next task:** Find out how to ensure lower semicontinuity of $\mathcal{F}$:

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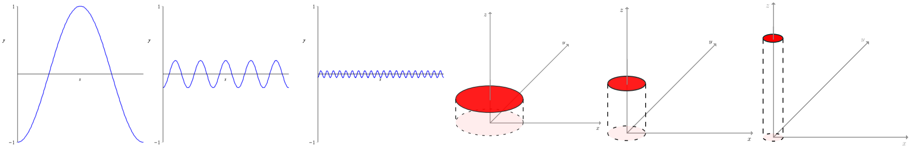
-A sequence (u_j) can only fail to converge area-strictly if (Du_j) exhibits either **oscillation and/or concentration** phenomenon:



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where

- ν_x is an $\mathbb{R}^m \times \mathbb{R}^{m\times d}$-measure valued map,
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-Young measures act as a functionals on f as follows:

$$\langle\langle f, \boldsymbol{\nu} \rangle\rangle := \int_{\Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times d}} f(x, y, A)\, \mathrm{d}\nu_x(y, A)\, \mathrm{d}x + \int_{\overline{\Omega} \times (\mathbb{R}^m \uplus \partial\mathbb{B}^m) \times \partial\mathbb{B}^{m \times d}} f^\infty(x, y, A)\, \mathrm{d}\nu_x^\infty(y, A)\, \mathrm{d}\lambda_\nu(x).$$

Figure: $u_j(x) = -1/j\cos(jx)$

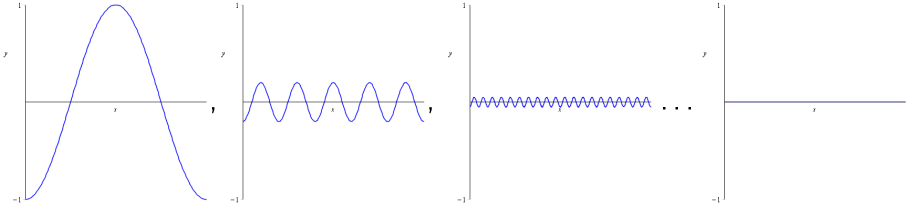
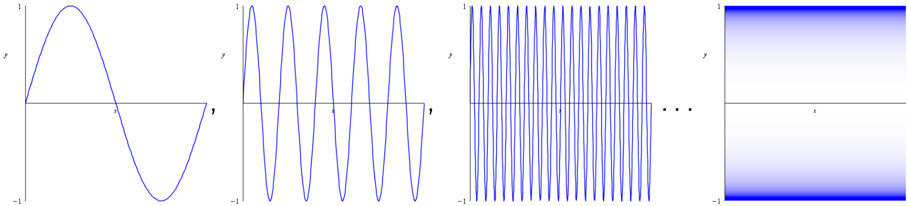


Figure: $\nabla u_j(x) = \sin(jx)$



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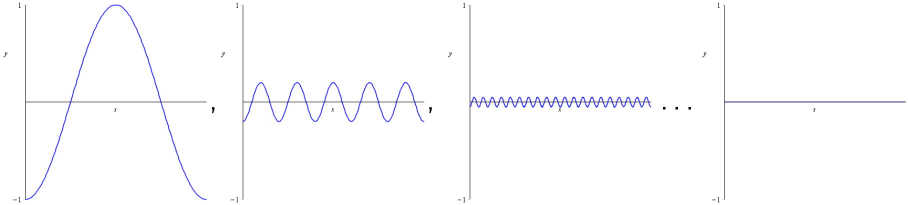

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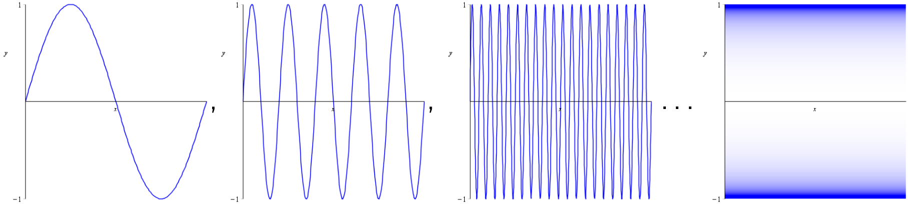


$$\nu_x(y, A) = \delta_0(y) \otimes \frac{1}{\pi\sqrt{1 - A^2}} \, \mathrm{d}A \llcorner [-1, 1], \ \lambda_\nu = 0.$$

Figure: $u_j(x) = \frac{j}{2\pi}\mathbb{1}_{\mathbb{D}}(jx_1, jx_2)$

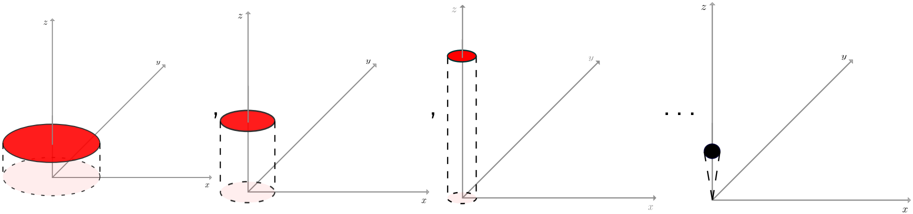


Figure: $Du_j(x) = \frac{j}{2\pi} x \mathcal{H}^1 \llcorner \mathbb{S}(0, \frac{1}{j})$

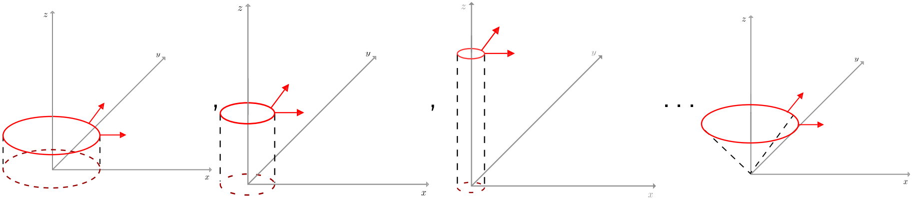
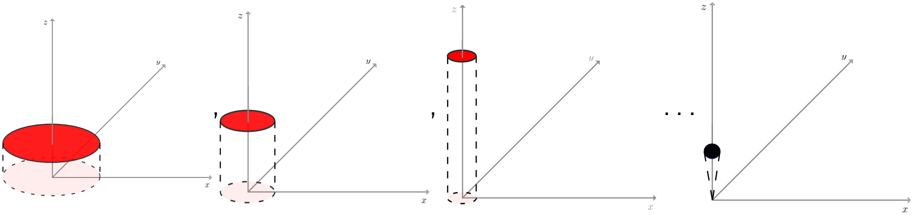


Figure: $u_j(x) = \frac{j}{2\pi}\mathbb{1}_{\mathbb{D}}(jx_1, jx_2)$



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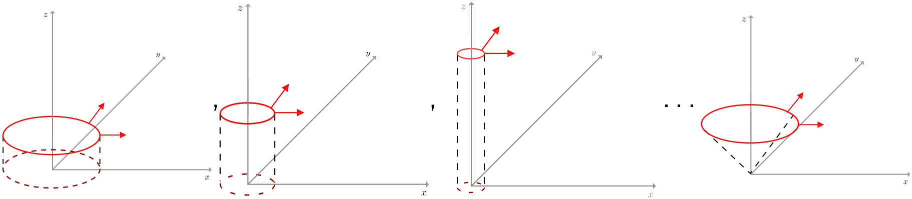

$$\nu_x(y, A) = \delta_0(y) \otimes \delta_0(A),$$

$$\lambda_\nu(x) = \delta_0(x), \quad \nu_x^\infty(y, A) = \delta_{+1}(y) \otimes \left(\frac{x}{2\pi}\mathcal{H}^1 \llcorner \mathbb{S}^1\right)(A).$$

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Theorem (Rindler and S., '14)

Let $(u_j) \subset BV(\Omega;\mathbb{R}^m)$ *be a norm bounded sequence. Then there exists a Young measure* $\boldsymbol{\nu}_u$ *and a (non relabelled) subsequence of* (u_j) *such that*

$$\mathcal{F}[u_j] \to \langle\!\langle f, \boldsymbol{\nu}_u \rangle\!\rangle \text{ as } j \to \infty$$

for every continuous f *which satisfies* $|f(x,y,A)| \leq C(1+|y|^p+|A|)$ *for some* $p \in [1, d/(d-1))$ *and is such that (the extended)* f^∞ *exists.*

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-**Thank you for listening!**