Dislocations: geometric and analytic properties (a joint work with N. Van Goethem)

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Introduction

We want to solve the minimum problem

$$\min_{(F,\mathcal{L})\in\mathcal{A}} \mathcal{W}(F,\Lambda_{\mathcal{L}}) = \mathcal{W}_e(F) + \mathcal{W}_{\text{defect}}(\Lambda_{\mathcal{L}}), \tag{1}$$

where $\mathcal{A}$ is the space of admissible couples of deformations and dislocation currents.

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where $\mathcal{A}$ is the space of admissible couples of deformations and dislocation currents. If $F_n \in L^p(\Omega,\mathbb{R}^{3\times 3})$ is a minimizing sequence, the policonvexity of $\mathcal{W}_e$ requires

$$M^{\beta}_{\bar{\alpha}}(F_n) \rightharpoonup M^{\beta}_{\bar{\alpha}}(F) \quad \text{weakly in} \quad L^p(\Omega), \tag{2}$$

for all multiindeces $\alpha,\ \beta$ such that $|\alpha|+|\beta|=3$.

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for all multiindeces $\alpha,\ \beta$ such that $|\alpha|+|\beta|=3$.
To prove this we have to focus on the finer properties of the functions F_n, using the fact that

$$-\operatorname{Curl} F_n = \Lambda^T_{\mathcal{L}_n},$$

for a bounded sequence of densities $\Lambda_{\mathcal{L}_n}$.

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Consider a function $F \in L^p(\Omega, \mathbb{R}^{3\times 3})$ satisfying

$$-\operatorname{Curl} F = b \otimes \mathcal{L},$$

for some dislocation (closed integral 1-current) $\mathcal{L}$. Since the Burgers vector b belongs to the lattice $\mathbb{Z}^3$ and $\mathcal{L}$ has integer multiplicity, whenever γ is a lipschitz closed curve, the path integral of F along γ turns out to be an element of $\mathbb{Z}^3$. We can then define GLOBALLY a deformation u whose gradient is F and whose value is univocally given up to an element of $\mathbb{Z}^3$.

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$\mathbb{T}^3$ being the 3-dimensional torus $\mathbb{R}^3/\mathbb{Z}^3$.
To prove convergence (2) we will use the fact that if $u_n \in W^{1,p}(\Omega, \mathbb{T}^3)$ are such that their graphs $\mathcal{G}_{u_n}$ satisfies $M(\partial \mathcal{G}_{u_n}) < C$ then there exists $u \in W^{1,p}(\Omega, \mathbb{T}^3)$ with (up to a subsequence)

$$M^{\beta}_{\bar{\alpha}}(Du_n) \rightharpoonup M^{\beta}_{\bar{\alpha}}(Du) \quad \text{weakly in} \quad L^p(\Omega),$$

for all $\alpha,\ \beta,\ |\alpha| + |\beta| = 3$.

Preliminaries

We introduce the torus $\mathbb{T} \cong \mathbb{R}/\sim$, where $a \sim b$ iff $a - b \in \mathbb{Z}$, and the standard projection $\pi_T : \mathbb{R} \to \mathbb{T}$. Now we will consider graphs of maps $u : \Omega \to \mathbb{T}^n$. These turn out to be n-rectifiable currents in $\Omega \times \mathbb{T}^n$.

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We set

$$\mathcal{D}^n(\Omega \times \mathbb{T}^n) := \{\omega \in \mathcal{D}^n(\Omega \times \mathbb{R}^n) : 1 - \text{periodic in the 2nd variable}\}.$$

If $\mathcal{T}$ is a n-current in $\mathcal{D}_n(\Omega \times \mathbb{R}^n)$ that has compact support in $\bar{\Omega} \times \mathbb{R}^n$, then it is well-defined the current $T(\mathcal{T}) \in \mathcal{D}_n(\Omega \times \mathbb{T}^n)$ defined as

$$T(\mathcal{T}) := T\llcorner_{\mathcal{D}^n(\Omega \times \mathbb{T}^n)}. \tag{3}$$

Moreover $M(T(\mathcal{T})) \leq M(\mathcal{T})$.
Let $u \in \mathcal{A}_p(\Omega, \mathbb{R}^n)$, then we define $T(u) : \Omega \to \mathbb{R}^n$ by $T(u) := \pi_T(u)$. It is easily seen that $T(\mathcal{G}_u)$ is a n-rectifiable current in $\Omega \times \mathbb{T}^n$.

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Definition

$$\mathcal{A}_p(\Omega, \mathbb{T}^n) := \{u \in L^p(\Omega, \mathbb{T}^n) : u \text{ is approx. diff. a.e. on } \Omega, \text{ and } M_{\bar{\alpha}}^{\beta}(Du) \in L^p(\Omega) \text{ for all } |\alpha| + |\beta| = n\} \tag{4}$$

$u \in \mathcal{A}_p(\Omega, \mathbb{T}^n) \Rightarrow \mathcal{G}_u$ is a n-rectifiable current in $\Omega \times \mathbb{T}^n$.

Preliminary results

Lemma

Let C be a smooth closed and simple curve in Ω and let S be a smooth surface in Ω bounded by C with unit normal N. The solution (up to a harmonic map on $\mathbb{R}^3$) of

$$\begin{cases} \Delta u = 0 & \text{in} \quad \mathbb{R}^3 \setminus S \\ [u] := u^+ - u^- = b & \text{on} \quad S \\ [\partial_N u] := \partial_N u^+ - \partial_N u^- = 0 & \text{on} \quad S \end{cases} \tag{5}$$

is given by

$$u(x) = -b \int_S \partial_N \Gamma(x' - x) dS(x'), \tag{6}$$

for $x \in \mathbb{R}^3 \setminus S$, where Γ is the solution in $\mathbb{R}^3$ of $\Delta\Gamma = \delta_0$.

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Moreover we can prove that there exists a constant $\gamma > 0$ independent of C and Ω, such that the solution u of (5) satisfies

$$|\partial_i u(x)| \leq \gamma \frac{b}{d(x, C)}. \tag{7}$$

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As a consequence of Lemma 5:

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Let $b \in \mathbb{Z}$*. Then the solution* u *of* (5) *belongs to* $C^\infty(\Omega \setminus C, \mathbb{T})$ *and it is harmonic in* $\Omega \setminus C$*.*

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As a consequence of Lemma 5:

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We consider a tubular neighborhood U of the curve C where we can use cylindrical coordinates and S coinciding with $\theta = 0$. We need this result of approximation:

Lemma

Let C and S as above and let u be the solution of the elliptic problem

$$\begin{cases} \Delta u = 0 & \text{on} \quad \mathbb{R}^3 \setminus S \\ u^+ - u^- = 1 & \text{on} \quad S \\ \partial_N^+ u - \partial_N^- u = 0 & \text{on} \quad S. \end{cases} \qquad (8)$$

Then for all $(\rho, \theta, z) \in U$ with $\theta \neq 0$ there exists the limit $\lim_{\epsilon \to 0^+} u(\epsilon\rho, \theta, z) = \theta + c$, where c is a fixed arbitrary constant.

Computation of the boundary

We denote by $\{e_1, e_2, e_3, \varepsilon_1, \varepsilon_2, \varepsilon_3\}$ the basis of $\Omega \times \mathbb{R}^3$.

Theorem

Let $b = (b_1, b_2, b_3) \in \mathbb{Z}^3$ and let $u = (u_1, u_2, u_3) : \Omega \to \mathbb{R}^3$ with u_i satisfying the equation (5) with $b = b_i$. Then $\mathcal{G}_u$ is the integral current in $\mathcal{D}_3(\Omega \times \mathbb{T}^3)$ and its boundary is given by

$$\partial \mathcal{G}_u(\omega) = -4\pi \mathcal{L} \otimes b(\varphi^\omega \circ \Phi), \tag{9}$$

for all $\omega \in \mathcal{D}^2(\Omega \times \mathbb{T}^3)$, where $\Phi := Id \times u$ and φ^ω is defined componentwise as

$$\varphi^\omega_{\alpha\beta} := \langle \omega, e_\alpha \wedge \varepsilon_\beta \rangle. \tag{10}$$

with $\alpha,\ \beta \in \{1, 2, 3\}$.

In words, φ^ω represents the tensor of the coefficients of the mixed components of ω.

Computation of the boundary

We procede by approximation. We set $\Omega_\epsilon := \Omega \setminus \bar{D}_\epsilon$, $u_\epsilon := u_{\llcorner \Omega_\epsilon}$, where $D_\epsilon := \{(\rho, \theta, z) \in U : \rho < \epsilon\}$. It is easy to see that $\mathcal{G}_{u_\epsilon} \rightharpoonup \mathcal{G}_u$ as current, so that we have

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u_ϵ is smooth in Ω_ϵ (Lemma 3) so we apply the Stokes theorem to compute the boundary of $\mathcal{G}_{u_\epsilon}$. For all 2-forms ω we have

$$\partial\mathcal{G}_{u_\epsilon}(\omega) = \int_{\partial D_\epsilon} \langle \omega \circ \Phi, \frac{\partial \Phi}{\partial \tau} \wedge \frac{\partial \Phi}{\partial x_3} \rangle dx = \int_{\partial D_\epsilon} \sigma_{ij}\omega_{ij}(x, u(x)) M^i_j(D(\Phi\llcorner_{\partial D_\epsilon}(x)))dx,$$

where (τ, x_3) is an orthogonal coordinate system in the tangent space to $\partial D_\epsilon = \partial\Omega_\epsilon$.

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where (τ, x_3) is an orthogonal coordinate system in the tangent space to $\partial D_\epsilon = \partial\Omega_\epsilon$. Explicitly

$$= \int_{\partial D_\epsilon} \omega_{23}(x, u(x))dx + \sum_{k=1}^{3} \int_0^{2\pi} \int_0^{l} \epsilon(\tilde{\omega}_{2k}(\epsilon, \theta, x_3, u(\epsilon, \theta, x_3)) \frac{\partial u_k}{\partial x_3}(\epsilon, \theta, x_3) dx_3 d\theta$$

$$- \sum_{k=1}^{3} \int_0^{l} \int_0^{2\pi} \epsilon\tilde{\omega}_{3k}(\epsilon, \theta, x_3, u(\epsilon, \theta, x_3)) \frac{\partial u_k}{\partial \tau}(\epsilon, \theta, x_3)) d\theta dx_3. \qquad (11)$$

Computation of the boundary

Integrations by parts and preceding lemmas provide

$$\sum_{k=1}^{3}\int_0^{2\pi}\epsilon\int_0^l(\frac{\partial\tilde{\omega}_{2k}}{\partial x_3}(\epsilon,\theta,x_3,u(\epsilon,\theta,x_3))u_k(\epsilon,\theta,x_3)dx_3d\theta =$$

$$\sum_{k=1}^{3}\int_0^{2\pi}\epsilon\int_0^l(\frac{\partial\tilde{\omega}_{2k}}{\partial x_3}(\epsilon,\theta,x_3,u(\epsilon,\theta,x_3))(\theta+o(1))dx_3d\theta,$$

where $o(1)\to 0$ as $\epsilon\to 0$, so that its absolute value can be estimated by $\|\frac{\partial\tilde{\omega}_{2k}}{\partial x_3}\|_\infty o(\epsilon)$, and thus also this term vanishes.

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As for the third term

$$= -\sum_{k=1}^{3}\int_0^{l}\tilde{\omega}_{3k}(0,\theta,x_3,b_1\theta,b_2\theta,b_3\theta)u_k(\epsilon,\theta,x_3)\Big|_0^{\theta=2\pi}dx_3$$

$$+\sum_{k=1}^{3}\int_0^{l}\int_0^{2\pi}\frac{d}{d\theta}\tilde{\omega}_{3k}(0,\theta,x_3,b_1\theta,b_2\theta,b_3\theta)u_k(\epsilon,\theta,x_3))d\theta dx_3+R(\epsilon). \qquad (12)$$

with $R(\epsilon)=o(1)$.

We then pass to the limit

$$-\sum_{k=1}^{3}\left(2\pi b_k\omega_{3k}(0,0,x_3,0,0,0)+\int_0^l\int_0^{2\pi}\frac{d}{d\theta}\tilde{\omega}_{3k}(0,0,x_3,b_1\theta,b_2\theta,b_3\theta)b_k\theta d\theta dx_3\right)$$

$$=-4\pi\sum_{k=1}^{3}b_k\int_0^l\omega_{3k}(0,0,x_3,b_1\theta,b_2\theta,b_3\theta)dx_3$$

$$=-4\pi\int_C(\tau(x)\otimes b)\cdot\varphi^\omega(x,u(x))d\mathcal{H}^1(x)=-4\pi\mathcal{L}\otimes b(\varphi^\omega\circ\Phi),$$

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and the proof is completed.
Moreover Theorem 5 can be generalized to an arbitrary closed integral 1-current $\mathcal{L}$, and it also holds

Lemma

Let S, C, b and u as in Theorem 5, and let $v\in C^1(\bar{\Omega},\mathbb{R}^3)$. Then $\mathcal{G}_{u+v}$ is an integral current in $\mathcal{D}_3(\Omega\times\mathbb{T}^3)$ and it holds

$$M(\partial\mathcal{G}_{u+v})\leq(1+24\sqrt{3}\pi\|Dv\|_{L^\infty(\Omega)})|\mathcal{L}\otimes b|(\Omega).\tag{13}$$

The minimum problem

Let us go back to the minimum problem

$$\min_{(F,\mathcal{L})\in\mathcal{A}} \mathcal{W}(F,\Lambda_{\mathcal{L}}), \tag{14}$$

where the energy $\mathcal{W}$ satisfies some appropriate convexity and coerciveness conditions (Van Goethem seminar), while $\mathcal{A}$ is the space of admissible couples of deformations and dislocation currents.

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For the minimizer we want to be satisfied the equilibrium condition:

$$-\,\mathrm{Div}\ \mathbb{C}\mathbb{E} = f \tag{15}$$

where f is an external force acting on the elastic body, $\mathbb{E} := \frac{1}{2}(F^T F - \mathrm{Id})$ is the Green St. Venant strain, and $\mathbb{C} := \dfrac{\partial^2 W_e}{\partial F_{ij}\partial F_{kl}}$ its associate elasticity tensor. Note that $f_i = -\dfrac{\partial^2 W_e}{\partial F_{ij}\partial F_{kl}}\partial_j F_{kl}$ and hence $f \otimes \mathbb{I} = \mathbb{C}\,\mathrm{Div}\ F$.

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We assume that W_e is smooth enough and that the material has finite compliance, that is, $\mathbb{C}^{-1}$ is finite. Moreover, for simplicity of discussion (but not of the model) we will assume $\mathbb{C}_{ijkl} = \delta_{ij}\delta_{kl}$. Hence it holds $f_i\delta_{jl} = \delta_{ij}(\operatorname{Div} F)_k$, that is ,

$$f_i = (\operatorname{Div} F)_i.$$

Existence result

So the class of admissible deformations is defined as follows. Let $f \in L^q(\Omega, \mathbb{R}^3)$ and let $b \in \mathbb{Z}^3$ a fixed Burgers vector, then

$$\mathcal{F}^b(f) := \{F \in L^p(\Omega, \mathbb{R}^{3\times 3}) : -\operatorname{Div} F = f \text{ and } -\operatorname{Curl} F = b \otimes \mathcal{L} \text{ for some closed integral 1-current } \mathcal{L}\}. \tag{16}$$

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The existence of a minimizer of $\mathcal{W}$ in $\mathcal{F}^b(f)$ is provided by the following:

Theorem

Let $p > 1$ and $f \in L^q(\Omega, \mathbb{R}^3)$ with $q > 3$. If $\mathcal{W}$ satisfies (...), then there exists a minimizer $F \in \mathcal{F}^b(f)$ of $\mathcal{W}$.

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A consequence of Helmoltz decomposition implies that, if $F \in \mathcal{F}^b(f)$, then F can be written as

$$F = Dv + Du + Dh,$$

where $v \in W_0^{2,q}(\Omega, \mathbb{R}^3)$ with $\Delta v = f$, $u \in W^{1,p}(\Omega, \mathbb{T}^3)$ is the function in Theorem 5, and $h \in C^\infty(\Omega, \mathbb{R}^3)$ is an harmonic map on Ω with some boundary conditions. The Sobolev embeddings Theorem provides $v \in C^1(\Omega, \mathbb{R}^3)$.

Sketch of the proof

To prove the existence result, we can use such decomposition for all the terms of a minimizing sequence $F_n = Dv_n + Du_n + Dh_n$. The coerciveness of the energy $\mathcal{W}_e$ gives a-priori bounds on the L^p norms of all the minors of Dv_n, Du_n, and Dh_n, while the coerciveness of $\mathcal{W}_{\text{defect}}$ gives an a-priori bound on the mass of the densities $\Lambda_{\mathcal{L}_n}$.
In particular there exist $F \in L^p(\Omega, \mathbb{R}^{3\times 3})$, $A \in L^p(\Omega, \mathbb{R}^{3\times 3})$, and $D \in L^p(\Omega)$ such that

$$F_k \rightharpoonup F \quad \text{weakly in } L^p(\Omega, \mathbb{R}^{3\times 3}), \tag{17a}$$

$$\mathrm{adj} F_k \rightharpoonup A \quad \text{weakly in } L^p(\Omega, \mathbb{R}^{3\times 3}), \tag{17b}$$

$$\det F_k \rightharpoonup D \quad \text{weakly in } L^p(\Omega). \tag{17c}$$

and a measure $\Lambda \in \mathcal{M}_b(\Omega, \mathbb{R}^{3\times 3})$ with

$$\Lambda_k \rightharpoonup \Lambda \quad \text{weakly* in } \mathcal{M}_b(\Omega, \mathbb{R}^{3\times 3}), \tag{18}$$

where we have set $\Lambda_k = \Lambda_{\mathcal{L}_k} = -\,\mathrm{Curl}\ F_k$

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and a measure $\Lambda \in \mathcal{M}_b(\Omega, \mathbb{R}^{3\times3})$ with

$$\Lambda_k \rightharpoonup \Lambda \quad \text{weakly* in } \mathcal{M}_b(\Omega, \mathbb{R}^{3\times3}), \tag{18}$$

where we have set $\Lambda_k = \Lambda_{\mathcal{L}_k} = -\,\mathrm{Curl}\ F_k$
Theorem 5 and Lemma 6 provide

$$M(\partial \mathcal{G}_{v_n+u_n+h_n}) < C,$$

thanks to the regularity of v_n and h_n.

Sketch of the proof

Then we can employ

Theorem

Let u_k be a sequence in $\mathcal{A}_p(\Omega, \mathbb{T}^3)$ such that $u_k \to u$ strongly in $L^p(\Omega, \mathbb{T}^3)$ and suppose that there exist functions $v^\alpha_\beta \in L^p(\Omega)$ such that $M^\beta_{\bar{\alpha}}(Du_k) \rightharpoonup v^\beta_\alpha$ for all multi-indices α and β with $|\alpha| + |\beta| = 3$. If

$$M(\partial \mathcal{G}_{u_k}) < C < +\infty \tag{19}$$

for all $k > 0$, then $u \in \mathcal{A}_p(\Omega, \mathbb{T}^3)$ and $v^\beta_\alpha = M^\beta_{\bar{\alpha}}(Du)$.

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to prove that $A = \mathrm{adj} F$, $D = \det F$.
The fact that there exists a closed integral 1-current $\mathcal{L}$ such that $\Lambda = \Lambda_{\mathcal{L}} = -\,\mathrm{Curl}\ F$ is an easy consequence of the compactness theorem for integral currents and of convergence (18). The conclusion then follows by lower semicontinuity.

Remarks and open questions

- We need the regularity condition of f. In general if u is as in Theorem 5 and v is another function, it is difficult to obtain estimates for the boundary of $\mathcal{G}_{u+v}$ better than (13).

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Thank you for attention!