Local and global minimality results for a nonlocal isoperimetric problem

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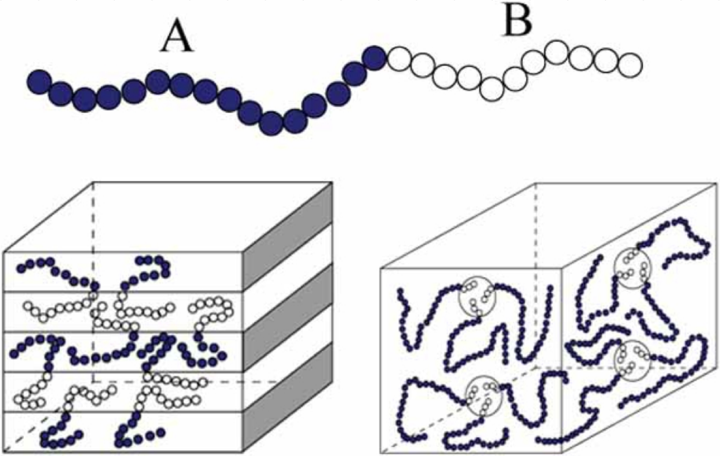
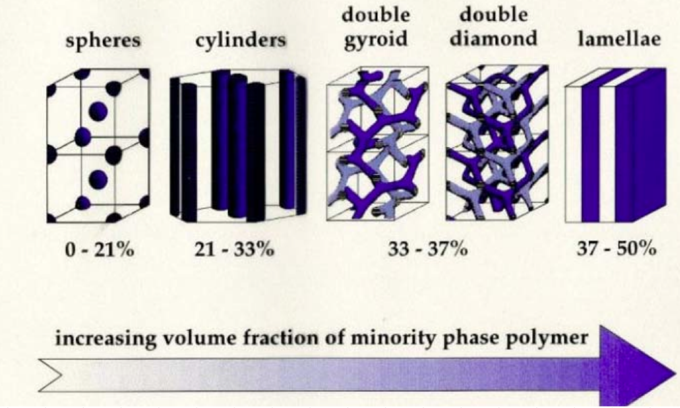
different volume fractions $\Longrightarrow$ different nanostructure

Phase boundaries for diblocks



Picture from an article by Choksi, Peletier and Willams 2009

Tendency towards *periodic* arrays of interfaces–
observed experimentally and numerically.



Picture from Edwin Thomas' talk at MSRI 1999

The Ohta-Kawasaki functional

Denote by $u : \Omega \to \mathbb{R}$ the function describing the density:

$$\begin{cases} u(x) \simeq 1 & \text{on phase } A \\ u(x) \simeq -1 & \text{on phase } B \end{cases} \qquad m = \frac{1}{|\Omega|} \int_\Omega u \, dx \qquad \text{fixed}$$

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and $\Omega = \mathbb{T}^N$ (PBC) or Ω is bounded and (sufficiently) smooth (NBC)

Setting

$$v(x) := \int_\Omega G_\Omega(x,y)u(y)\,dy\,,$$

we have that v solves

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Note that

$$NL(u) = \int_\Omega \int_\Omega G_\Omega(x,y)u(y)u(x)\,dydx = \int_\Omega v(x)u(x)\,dx$$

$$= \int_\Omega v(x)\underbrace{\big(u(x)-m\big)}_{=-\Delta v}\,dx$$

$$= \int_\Omega |\nabla v|^2\,dx$$

Remark: We have that

$$\|\nabla v\|_{L^2(\Omega)} = \sup\Big\{\int_\Omega (u-m)\varphi\,dx : \varphi\in H^1(\Omega)\,,\|\nabla\varphi\|_{L^2(\Omega)}=1\Big\}$$

Hence NL may be regarded as the square of the homogeneous H^{-1}-norm of $u-m$.

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The nonlocal isoperimetric problem–the $\varepsilon \to 0$ limit of O-K

The nonlocal isoperimetric problem (NLIP) is given by

minimize $\quad J(u) := \mathrm{Per}_\Omega(\{x : u(x) = 1\}) + \gamma \int_\Omega |\nabla v|^2 \, dx, \quad (NLIP)$

over all $u \in BV(\Omega, \{\pm 1\})$ satisfying

$$\fint_\Omega u \, dx = m$$

and v satisfying

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NLIP– A ‘fight’ of low surface area versus high oscillation

$$J(u) := \underbrace{\mathrm{Per}_\Omega(\{x \,:\, u(x) = 1\})}_{\text{attractive short-range interactions}} + \gamma \underbrace{\int_\Omega \int_\Omega G_\Omega(x,y)u(x)u(y)\,dxdy}_{\text{repulsive long-range interactions}}$$

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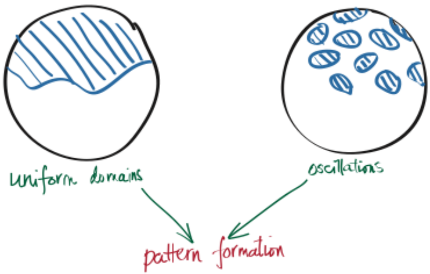

NLIP is interesting as a canonical nonlocal perturbation of the classical isoperimetric problem; and as a model for energy-driven pattern formation.

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We rewrite the functional in a more geometric fashion:

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$$\min \{ J(E) : \ E \subset \Omega \, , |E| = m \} \qquad \text{(NLIP)}$$

Mathematical challenges: global minimizers

Structure of global minimizers: are the global minimizers of $\mathcal{E}_\varepsilon$ and of J (almost) periodic?

True in one dimension (Müller, 1993);

Partial results in higher dimensions (Alberti-Choksi-Otto, 2009; Spadaro, 2009) $\rightsquigarrow$ equidistribution of energy

Droplet minimizers: Choksi&Peletier, Muratov, Goldman Muratov&Serfaty, Cicalese&Spadaro

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Link between local minimizers of $\mathcal{E}_\varepsilon$ and J

The link is provided via Γ-convergence

Theorem (Kohn-Sternberg)

Let (X, d) be a metric space and let $(F_k)_k$ be a sequence of lower semicontinuous and equi-coercive functionals from X to $\mathbb{R} \cup \{+\infty\}$. Assume that $F_k \xrightarrow{\Gamma(d)} F$ and let x be an isolated local minimizer of F. Then there exist $x_k \to x$ such that x_k is a local minimizers of F_k for k large enough.

Plan:

- show that $\partial^2 J > 0 \Rightarrow$ a critical point is an isolated L^1-local minimizer;
- thus, the known stable critical configurations are isolated L^1-local minimizers;
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The Euler-Lagrange equation for C^2 minimizers of $J(E)$ with a volume constraint

$$(E.L.) \qquad H_{\partial E}(x) + 4\gamma v_E(x) = \lambda \qquad \text{on } \partial E$$

where $H_{\partial E}$ = sum of principal curvatures

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Natural definition: $E \subset \Omega$, is a (strict) local minimizer if $\exists \delta > 0$ s.t.

$$J(F) > J(E)$$

whenever $F \subset \Omega$ with $0 < d(E,F) < \delta$ and $|F| = |E|$

Easy fact:

$$\left|\int_\Omega |\nabla v_E|^2\,dx - \int_\Omega |\nabla v_F|^2\,dx\right| \leq c|E\Delta F|$$

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Theorem (Sternberg-Topaloglu (2011))

If $E \subset \mathbb{T}^n$ is a local minimizer of J, then $\partial E \setminus \Sigma$ is $C^{3,\alpha}$, for any $\alpha < 1$, and Σ is a closed set such that $\dim_{\mathcal{H}}(\Sigma) \leq n - 8$

Second variation

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Let $E \in C^2$ be critical and fix a C^2 vector field $X : \Omega \mapsto \Omega$. Then, let us consider

$\Phi : \Omega \times (-1, 1) \mapsto \Omega$ the associated flow

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Assume that

$$|E_t| = |E|.$$

Computation of the second variation

Theorem (Choksi-Sternberg 2007)

For, say $\Omega = \mathbb{T}^n$ *and* $f := X \cdot \nu_E$, *one can compute second variation*

$$\partial^2 J(f) := \underbrace{\int_{\partial E} \left(|\nabla_{\partial E} f|^2 - |B|^2 f^2 \right) d\mathcal{H}^{n-1}}_{\text{standard second variation of area}} +$$

$$\underbrace{8\gamma \int_{\partial E} \int_{\partial E} G(x,y)\, f(x)\, f(y)\, d\mathcal{H}^{n-1}_x\, d\mathcal{H}^{n-1}_y}_{\text{nonlocal contribution}}$$

$$+ \underbrace{4\gamma \int_{\partial E} \nabla v \cdot \nu\, f^2\, d\mathcal{H}^{n-1}}_{\text{nonlocal contribution}}.$$

Here, $G(x,y)$=*Green's function for (–) Laplacian.*

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Strict stability then means

$$\partial^2 J(E)[f] > 0 \quad \forall f \in T^\perp \setminus \{0\}$$

L^1 local minimality via second variation

Theorem (Acerbi-Fusco-M., Comm. Math. Phys 2013)

Let $E \subset \mathbb{T}^n$ be a regular critical set of J such that

$$\partial^2 J(E)[f] > 0 \qquad \forall f \in T^{\perp}(\partial E) \setminus \{0\} .$$

There exists $\delta > 0$ s.t. for all $F \subset \mathbb{T}^n$ with $|F| = |E|$ and $d(E,F) < \delta$

$$J(F) \geq J(E) + C_0 d(E,F)^2 .$$

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- Extension to NBC: Julin-Pisante, 2013

Consequences: the Ohta-Kawasaki energy

Proposition

If E is a regular critical point of J with $\partial^2 J(E) > 0$, then there exists a family $\{u_\varepsilon\}_{\varepsilon<\varepsilon_0}$ of strict local minimizers of the diffused energy $\mathcal{E}_\varepsilon$ with $\int_{\mathbb{T}^n} u_\varepsilon \, dx = \int_{\mathbb{T}^n} u_E \, dx$, s.t.

$$u_\varepsilon \to u_E \qquad \text{in} \quad L^1(\mathbb{T}^n)$$

as $\varepsilon \to 0$.

Case $\gamma = 0$: the periodic isoperimetric problem

Corollary

Let $E \subset \mathbb{T}^n$ be smooth open set with ∂E a constant mean curvature surface. If

$$\int_{\partial E} \left(|\nabla_{\partial E} f|^2 - |B_{\partial E}|^2 f^2\right) d\mathcal{H}^{n-1} > 0 \qquad \forall\, f \in T^{\perp}(\partial E) \setminus \{0\}\,,$$

then there exist δ, $C > 0$ s.t. for $F \subset \mathbb{T}^n$, with $|F| = |E|$ and $d(E, F) < \delta$

$$P_{\mathbb{T}^n}(F) \geq P_{\mathbb{T}^n}(E) + C[d(E, F)]^2\,.$$

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- B.White, 1994: local minimality w.r.t. L^∞ perturbations

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- Morgan-Ros, 2010: local minimality w.r.t. L^1 perturbations but for $n \leq 7$ and without quantitative estimate

Application: Global minimality of the single lamella

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$$(\mathcal{P}) \qquad \text{Min} \left\{ J_\gamma(E) = P_{\mathbb{T}^n}(E) + \gamma \int_{\mathbb{T}^n} |\nabla v_E|^2 \, dx \,, \qquad |E| = d \right\}$$

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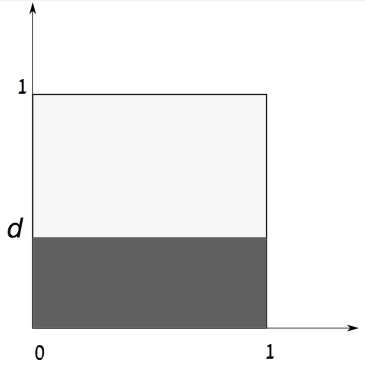
For $0 < d < 1$ set

$$L = \mathbb{T}^{n-1} \times [0, d]$$



Global minimality of the single lamella

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Theorem (Acerbi-Fusco-M.)

Assume that L is the unique, up to translations and relabelling of coordinates, global minimizer of the periodic isoperimetric problem. Then L is also the unique global minimizer of ($\mathcal{P}$), provided γ is sufficiently small.

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Step 4: For γ small L is an L^1 isolated local minimizer, uniformly in γ. Thus $E_\gamma = L$ for γ small.

Global minimality of the single lamella

Let $n = 2$. Theorem + Howards-Hutchings-Morgan, 1999

$$\Downarrow$$

If $\frac{1}{\pi} < d < 1 - \frac{1}{\pi}$, L is the unique global minimizer of $(\mathcal{P})$ in $\mathbb{T}^2$ provided γ is small (see also Sternberg-Topaloglu, 2011)

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Let $n = 3$. Theorem + a result of Hadwiger, 1972

$$\Downarrow$$

There exist $\varepsilon_0, \gamma_0 > 0$ s.t. if $\frac{1}{2} - \varepsilon_0 < d < \frac{1}{2} + \varepsilon_0$, $0 \leq \gamma < \gamma_0$ L is the unique global minimizer of $(\mathcal{P})$ in $\mathbb{T}^3$

Critical 2d k-lamellar patterns

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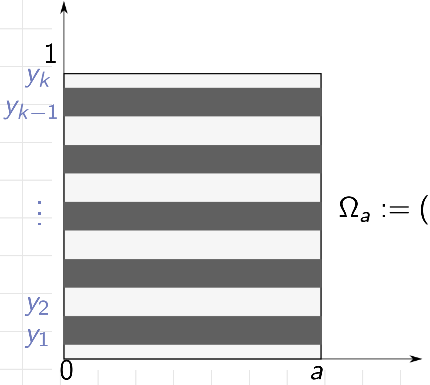

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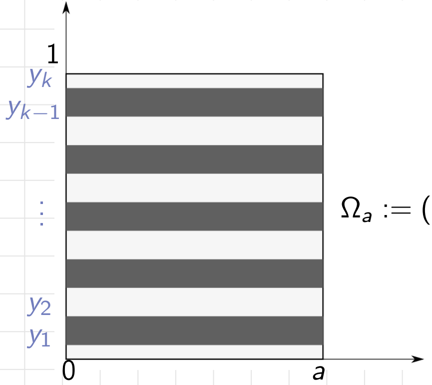

$\Omega_a := (0, a) \times (0, 1)$

- Criticality $\Rightarrow y_j = \frac{2j-1}{2k} \qquad j = 1, \ldots, k$

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then the k-lamellar critical point u_k is an isolated L^1-local minimizer in $\Omega_a := (0, a) \times (0, 1)$; i.e., there exist δ and $C > 0$ such that

$$J(w) \geq J(u_k) + C \, \|u_k - w\|^2_{L^1(\Omega_a)}$$

provided $\|u_k - w\|_{L^1(\Omega_a)} < \delta$ and $\int_{\Omega_a} w = \int_{\Omega_a} u_k$.

Local minimality of 2d lamellar patterns: ingredients

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- Spectral analysis: $a < \pi\sqrt{\frac{k}{2\gamma}}$ implies that

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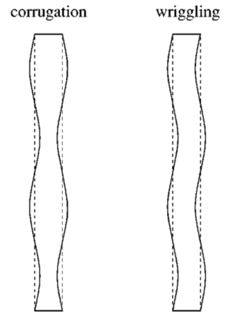
- Remark: Morphological instability may only come from wriggling and corrugations:



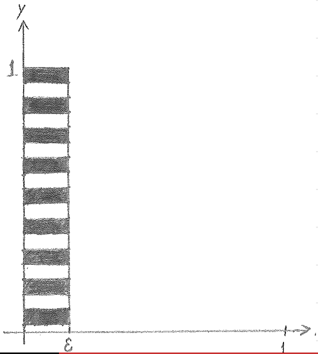
Cascade of lamellar global minimizers in thin rectangles

Theorem (M.-Sternberg, 2013)

Fix any $\gamma > 0$. *Then for* $k(\gamma)$ *minimizing the 1d energy*

$$J^{1d}(u_k) = k + \frac{\gamma}{12k^2},$$

$u_{k(\gamma)}$ *is the unique global minimizer of* J *in* $(0,\varepsilon) \times (0,1)$ *for* ε *sufficiently small.*



Step 1: Γ-convergence

Consider (NLIP) posed on a thin rectangle $\Omega_\varepsilon := (0,\varepsilon)\times(0,1)$:

$$\min \quad J_{\Omega_\varepsilon}(u) := \mathrm{Per}_{\Omega_\varepsilon}(\{x:\ u(x)=1\}) + \gamma\int_{\Omega_\varepsilon} |\nabla v|^2\,dx\,,$$

under the constraint $\int_{\Omega_\varepsilon} u = 0$.

After rescaling onto the (fixed) unit square Ω_1 and dividing by ε one checks that J_{Ω_ε} Γ-converges as $\varepsilon\to 0$ to the one-dimensional energy

$$J^{1d}(u) := \#\text{ of jumps } + \gamma\int_0^1 v_y^2\,dy$$

for $u\in BV((0,1);\{\pm1\})$, with $\int_0^1 u(y)\,dy = 0$, and where $v=v(y)$ solves

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"Perimeter" in 1d just counts number of jumps, located say at $\{y_j\}_{j=1}^{k}$.
Also criticality condition $\kappa + 4\gamma v = constant$ reduces to simply

$$v(y_j) = constant \quad \text{(since curvature } = 0)$$

Easily check that only critical points are u_k given by

$$u_k(y) := \begin{cases} 1 & \text{for} \quad 0 < y < y_1,\ y_2 < y < y_3, \ldots, y_{k-1} < y < y_k \\ -1 & \text{for} \quad y_1 < y < y_2,\ y_3 < y < y_4, \ldots, y_k < y < 1 \end{cases}$$

where $y_j := \frac{2j-1}{2k}$ for $j = 1, 2, \ldots, k$ and the derivative of the corresponding v_k is a "sawtooth function."
(For simplicity only, fix $m = 0$.)

Step 2: Explicit calculation of 1d minimizers

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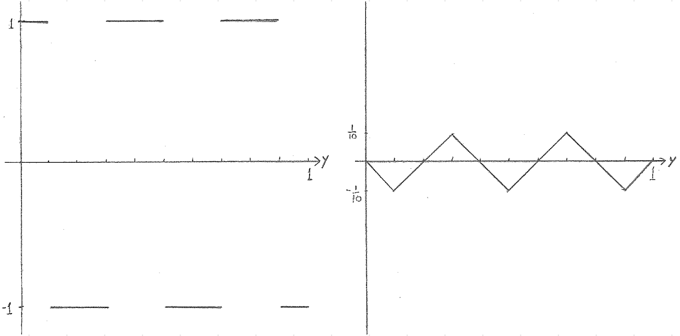

The 5-jump critical point u_5 with $m = 0$ and the derivative of the corresponding v_5 solving

$$-v_5'' = u_5 \quad \text{for } 0 < y < 1, \quad v_5'(0) = 0 = v_5'(1).$$

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Easy to compute that

$$J^{1d}(u_k) = k + \frac{\gamma}{12k^2}$$

so given any γ, one can optimize over k to find global minimizer is $u_{k(\gamma)}$, with

$$k(\gamma) \sim \gamma^{1/3} \text{ for } \gamma \gg 1.$$

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Step 3: Proof of global minimality

Fix $a < \pi\sqrt{\frac{k(\gamma)}{2\gamma}}$. Let u_ε be a global minimizer in Ω_ε and let

$$\tilde{u}_\varepsilon(x, y) := u_\varepsilon(\tfrac{\varepsilon}{a}x, y)$$

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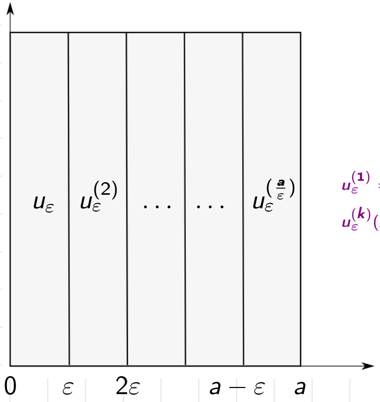
Let u_ε^r the function in $BV(\Omega_a)$ obtained by $\frac{a}{\varepsilon} - 1$ reflections of u_ε.



$u_\varepsilon^{(1)} = u_\varepsilon$

$u_\varepsilon^{(k)}(x,y) := u^{(k-1)}(x, 2(k-1)\varepsilon - y)$

Step 3: Proof of global minimality: conclusion

Note

$$\|u^r_\varepsilon - u_{k(\gamma)}\|_{L^1(\Omega_a)} = \|\tilde{u}_\varepsilon - u_{k(\gamma)}\|_{L^1(\Omega_a)} \to 0$$

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$$\Downarrow$$

$u_\varepsilon = u_{k(\gamma)}$ for ε small

Thanks for your attention!!!!!