Local and global minimality results for a nonlocal isoperimetric problem

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Diblock copolymers

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Chemical incompatibility between the non-affine subchains
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Chemical incompatibility between the phases \(\implies\) tendency to macroscopic phase separation
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\[
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Chemical bonds between the non-affine subchains $\implies$ long range effects, pattern formation
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\[ \cdots A\cdots A\cdots A\cdots A\cdots A\cdots A\cdots A\cdots B\cdots B\cdots B\cdots B\cdots B\cdots B\cdots B\cdots \cdots \]

Chemical incompatibility between the phases $\implies$ tendency to macroscopic phase separation

Chemical bonds between the non-affine subchains $\implies$ long range effects, pattern formation

different volume fractions $\implies$ different nanostructure
Phase boundaries for diblocks

Picture from an article by Choksi, Peletier and Williams 2009
Tendency towards periodic arrays of interfaces—observed experimentally and numerically.

Picture from Edwin Thomas’ talk at MSRI 1999
The Ohta-Kawasaki functional

Denote by \( u : \Omega \rightarrow \mathbb{R} \) the function describing the density:

\[
\begin{cases}
  u(x) \approx 1 & \text{on phase } A \\
  u(x) \approx -1 & \text{on phase } B
\end{cases}
\]

\[ m = \frac{1}{|\Omega|} \int_{\Omega} u \, dx \quad \text{fixed} \]
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\[
\mathcal{E}_\varepsilon(u) = \varepsilon \int_{\Omega} |\nabla u|^2 \, dx + \frac{1}{\varepsilon} \int_{\Omega} (1 - u^2)^2 \, dx + \gamma NL(u)
\]

Here \( G_\Omega \) is the Green's function defined by

\[
\begin{align*}
    -\Delta y \, G_\Omega(x, \cdot) &= \delta_{x-1} \quad \text{in } \Omega, \\
    \int_{\Omega} G_\Omega(x, y) \, dy &= 0
\end{align*}
\]

PBC or NBC and \( \Omega = T^N(PBC) \) or \( \Omega \) is bounded and (sufficiently) smooth (NBC).
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where

\[
\text{NL}(u) := \int_{\Omega} \int_{\Omega} G_\Omega(x, y)u(x)u(y) \, dx \, dy.
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and \( \Omega = \mathbb{T}^N \) (PBC) or \( \Omega \) is bounded and (sufficiently) smooth (NBC)
Setting

\[ v(x) := \int_{\Omega} G_\Omega(x, y) u(y) \, dy , \]

we have that \( v \) solves

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-\Delta v = u - m & \text{in } \Omega, \\
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\]

where \( m \) is the prescribed average of \( u \)

\[ m := \int_\Omega u \, dx . \]
The Ohta-Kawasaki functional: the potential \( \nu \)

Setting

\[
\nu(x) := \int_{\Omega} G_\Omega(x, y) u(y) \, dy ,
\]

we have that \( \nu \) solves

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\begin{cases}
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Note that

\[ NL(u) = \int_{\Omega} \int_{\Omega} G_{\Omega}(x, y) u(y) u(x) \, dy \, dx = \int_{\Omega} v(x) u(x) \, dx \]

\[ = \int_{\Omega} v(x) (u(x) - m) \, dx \]

\[ = \int_{\Omega} |\nabla v|^2 \, dx \]

Remark: We have that

\[ \| \nabla v \|_{L^2(\Omega)} = \sup \left\{ \int_{\Omega} (u - m) \phi \, dx : \phi \in H^1(\Omega), \| \nabla \phi \|_{L^2(\Omega)} = 1 \right\} \]

Hence \( NL \) may be regarded as the square of the homogeneous \( H^{-1} \)-norm of \( u - m \).
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The Ohta-Kawasaki functional: the nonlocal term revisited

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Hence \( NL \) may be regarded as the square of the homogeneous \( H^{-1} \)-norm of \( u - m \).
The nonlocal isoperimetric problem—the $\varepsilon \to 0$ limit of O-K

The nonlocal isoperimetric problem (NLIP) is given by

$$\min J(u) := \text{Per}_\Omega \left( \{ x : u(x) = 1 \} \right) + \gamma \int_\Omega |\nabla v|^2 \, dx,$$

over all $u \in BV(\Omega, \{\pm 1\})$ satisfying

$$-\int_\Omega udx = m$$

and $v$ satisfying

$$-\Delta v = u - m \quad \text{in} \quad \Omega,$$

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$G_\Omega(x, y) \sim \frac{1}{|x-y|^{N-2}}$

**attractive short-range interactions**

**repulsive long-range interactions**
NLIP—A ‘fight’ of low surface area versus high oscillation

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- \text{attractive short-range interactions}
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NLIP is interesting as a canonical nonlocal perturbation of the classical isoperimetric problem; and as a model for energy-driven pattern formation.
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and \( u_E := \chi_E - \chi_{E^c} \).
The Nonlocal Isoperimetric Problem (NLIP)

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\[ \min \{ J(E) : E \subset \Omega, |E| = m \} \quad (\text{NLIP}) \]
Structure of global minimizers: are the global minimizers of $E_\varepsilon$ and of $J$ (almost) periodic?

True in one dimension (Müller, 1993);

Partial results in higher dimensions (Alberti-Choksi-Otto, 2009; Spadaro, 2009)

$\Rightarrow$ equidistribution of energy

Droplet minimizers: Choksi&Peletier, Muratov, Goldman Muratov&Serfaty, Cicalese&Spadaro

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Mathematical challenges: global minimizers

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Structure of local minimizers: Can we exhibit local minimizers of $E_\varepsilon$ and of $J$ that look like lamellae, droplets, gyroids, cylinders....? Can we provide sufficiency conditions for local minimality?

Choksi & Sternberg, 2007: computation of $\partial^2 J$ at critical points

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The link is provided via $\Gamma$-convergence

**Theorem (Kohn-Sternberg)**

Let $(X, d)$ be a metric space and let $(F_k)_k$ be a sequence of lower semicontinuous and equi-coercive functionals from $X$ to $\mathbb{R} \cup \{+\infty\}$. Assume that $F_k \xrightarrow{\Gamma(d)} F$ and let $x$ be an isolated local minimizer of $F$. Then there exist $x_k \to x$ such that $x_k$ is a local minimizer of $F_k$ for $k$ large enough.

Plan:

- show that $\partial^2 J > 0 \Rightarrow$ a critical point is an isolated $L^1$-local minimizer;
- thus, the known stable critical configurations are isolated $L^1$-local minimizers;
- by Kohn&Sternberg theory such configurations can be lifted to locally minimizing configurations for the diffuse energy $E_\varepsilon$. 
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The Euler-Lagrange equation for \( C^2 \) minimizers of \( J(E) \) with a volume constraint

\[(E.L.) \quad H_{\partial E}(x) + 4\gamma v_E(x) = \lambda \quad \text{on } \partial E\]

where \( H_{\partial E} = \text{sum of principal curvatures} \)
Distance between (equivalence classes) of sets:

- **NBC case:** $d(E, F) = |E \triangle F|$.
- **PBC case:** $d(E, F) = \min_{\tau} |E \triangle (F + \tau)|$.

Natural definition: $E \subset \Omega$, is a (strict) local minimizer if $\exists \delta > 0$ s.t. $J(F) > J(E)$ whenever $F \subset \Omega$ with $0 < d(E, F) < \delta$ and $|F| = |E|$. 
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Regularity of local minimizers

Easy fact:

\[ \left| \int_{\Omega} |\nabla v_E|^2 \, dx - \int_{\Omega} |\nabla v_F|^2 \, dx \right| \leq c |E \Delta F| \]
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Another fact:

\[ E \text{ is a minimizer with volume constraint} \iff \]
\[ E \text{ is a minimizer of } J(F) + \Lambda ||F| - |E|| \text{ for some large } \Lambda \]
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Theorem (Sternberg-Topaloglu (2011))

If \( E \subset \mathbb{T}^n \) is a local minimizer of \( J \), then \( \partial E \setminus \Sigma \) is \( C^{3,\alpha} \), for any \( \alpha < 1 \), and \( \Sigma \) is a closed set such that \( \dim_{\mathcal{H}}(\Sigma) \leq n - 8 \)
Let $E \in C^2$ be critical and fix a $C^2$ vector field $X : \Omega \mapsto \Omega$. Then, let us consider $\Phi : \Omega \times (-1, 1) \mapsto \Omega$ the associated flow $\frac{\partial \Phi}{\partial t} = X(\Phi)$, $\Phi(x, 0) = x$ and set $E_t := \Phi(\cdot, t)(E)$.

Assume that $|E_t| = |E|$.
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Assume that

$$|E_t| = |E|.$$
Theorem (Choksi-Sternberg 2007)

For, say $\Omega = \mathbb{T}^n$ and $f := X \cdot \nu_E$, one can compute second variation

$$\partial^2 J(f) := \int_{\partial E} \left( |\nabla_E f|^2 - |B|^2 f^2 \right) d\mathcal{H}^{n-1} +$$

\begin{align*}
\text{standard second variation of area} & \\
8\gamma \int_{\partial E} \int_{\partial E} G(x, y) f(x) f(y) d\mathcal{H}^{n-1}_x d\mathcal{H}^{n-1}_y & \\
\text{nonlocal contribution} & \\
+ 4\gamma \int_{\partial E} \nabla v \cdot \nu f^2 d\mathcal{H}^{n-1} & \\
\text{nonlocal contribution} & \\
\end{align*}

Here, $G(x, y) =$ Green’s function for $(-)$ Laplacian.
Consequences of the translation invariance

Since \( J(E) = J(E + t\tau) \)
Consequences of the translation invariance

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$$\iff \partial^2 J(E)[\tau \cdot \nu_E(x)] = 0 \quad \text{for all } \tau$$
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Since \( J(E) = J(E + t\tau) \)

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Let us define \( T = \text{span}\{\nu_1, \ldots, \nu_n\} \) and

\[ T^\perp = \{ f \in H^1(\partial E) : \int f = \int f \nu_i = 0 \} \]
Consequences of the translation invariance

Since \( J(E) = J(E + t\tau) \)

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Strict stability then means

\[ \partial^2 J(E)[f] > 0 \quad \forall f \in T^\perp \setminus \{0\} \]

Let $E \subset \mathbb{T}^n$ be a regular critical set of $J$ such that

$$\partial^2 J(E)[f] > 0 \quad \forall f \in T^\perp(\partial E) \setminus \{0\}.$$

There exists $\delta > 0$ s.t. for all $F \subset \mathbb{T}^n$ with $|F| = |E|$ and $d(E, F) < \delta$

$$J(F) \geq J(E) + C_0 d(E, F)^2.$$

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- Extension to NBC: Julin-Pisante, 2013
Proposition

If $E$ is a regular critical point of $J$ with $\partial^2 J(E) > 0$, then there exists a family $\{u_\varepsilon\}_{\varepsilon < \varepsilon_0}$ of strict local minimizers of the diffused energy $E_\varepsilon$ with $\int_{\mathbb{T}^n} u_\varepsilon \, dx = \int_{\mathbb{T}^n} u_E \, dx$, s.t.

$$u_\varepsilon \to u_E \quad \text{in} \quad L^1(\mathbb{T}^n)$$

as $\varepsilon \to 0$. 
Case $\gamma = 0$: the periodic isoperimetric problem

**Corollary**

Let $E \subset \mathbb{T}^n$ be smooth open set with $\partial E$ a constant mean curvature surface. If

$$\int_{\partial E} \left( |\nabla \partial E f|^2 - |B_{\partial E}|^2 f^2 \right) d\mathcal{H}^{n-1} > 0 \quad \forall \, f \in T^\perp(\partial E) \setminus \{0\},$$

then there exist $\delta, C > 0$ s.t. for $F \subset \mathbb{T}^n$, with $|F| = |E|$ and $d(E, F) < \delta$

$$P_{\mathbb{T}^n}(F) \geq P_{\mathbb{T}^n}(E) + C[d(E, F)]^2.$$
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- B. White, 1994: local minimality w.r.t. $L^\infty$ perturbations
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- B. White, 1994: local minimality w.r.t. $L^\infty$ perturbations
- Morgan-Ros, 2010: local minimality w.r.t. $L^1$ perturbations but for $n \leq 7$ and without quantitative estimate
Application: Global minimality of the single lamella

\[ \min \{ J(\gamma(E)) = P_Tn(E) + \gamma \int T_n |\nabla v_E|^2 \text{dx}, |E| = d \} \]

For \(0 < d < 1\) set \(L = T_n - 1 \times [0, d]\)
Application: Global minimality of the single lamella

\[(P) \quad \text{Min} \left\{ J_\gamma(E) = P_{\mathbb{T}^n}(E) + \gamma \int_{\mathbb{T}^n} |\nabla v_E|^2 \, dx, \quad |E| = d \right\} \]
(P) \quad \text{Min} \begin{cases} J_\gamma(E) = P_{\mathbb{T}^n}(E) + \gamma \int_{\mathbb{T}^n} |\nabla v_E|^2 \, dx, & |E| = d \end{cases}

For \(0 < d < 1\) set

\[ L = \mathbb{T}^{n-1} \times [0, d] \]
Global minimality of the single lamella

Theorem (Acerbi-Fusco-M.) Assume that $L$ is the unique, up to translations and relabelling of coordinates, global minimizer of the periodic isoperimetric problem. Then $L$ is also the unique global minimizer of $(P)$ provided $\gamma$ is sufficiently small.

Sketch of the proof: Let $E_\gamma$ be a minimizer of $J_\gamma$.

Step 1: As $\gamma \to 0$ we have $E_\gamma \to L$ in $L^1$.

Step 2: For $\gamma$ small $L$ is strictly stable for $J_\gamma$.

Step 4: For $\gamma$ small $L$ is an $L^1$ isolated local minimizer, uniformly in $\gamma$. Thus $E_\gamma = L$ for $\gamma$ small.
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Global minimality of the single lamella

Let $n = 2$. Theorem + Howards-Hutchings-Morgan, 1999

⇓

If $\frac{1}{\pi} < d < 1 - \frac{1}{\pi}$, $L$ is the unique global minimizer of $(\mathcal{P})$ in $\mathbb{T}^2$

provided $\gamma$ is small (see also Sternberg-Topaloglu, 2011)
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Let $n = 3$. Theorem + a result of Hadwiger, 1972

$\Downarrow$

There exist $\varepsilon_0, \gamma_0 > 0$ s.t. if

$\frac{1}{2} - \varepsilon_0 < d < \frac{1}{2} + \varepsilon_0$, $0 \leq \gamma < \gamma_0$

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We consider only the case $m = 0$
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$$\Omega_a := (0, a) \times (0, 1)$$
Critical 2d $k$-lamellar patterns

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$\bullet$ Criticality $\Rightarrow y_j = \frac{2j-1}{2k}$, $j = 1, \ldots, k$
Theorem (M.-Sternberg, 2013)

For any positive integer $k$,
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For any positive integer $k$, if

$$a < \pi \sqrt{\frac{k}{2\gamma}},$$

then the $k$-lamellar critical point $u_k$ is an isolated $L^1$-local minimizer in $\Omega := (0, a) \times (0, 1)$; i.e., there exist $\delta$ and $C > 0$ such that

$$J(w) \geq J(u_k) + C \|u_k - w\|_{L^1(\Omega)}$$

provided $\|u_k - w\|_{L^1(\Omega)} < \delta$ and $\int_{\Omega} w = \int_{\Omega} u_k$. 

Local minimality of 2d lamellar patterns
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Local minimality of 2d lamellar patterns

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Local minimality of 2d lamellar patterns: ingredients

- Spectral analysis: $\sqrt{k^2} < \pi$ implies that $\partial^2 J(u_k)[f] > 0$ for all $f \in H^1_0(\Gamma)$ with $\int_\Gamma f \, dH_1 = 0$.

- Extension of the Acerbi-Fusco-M. $\partial^2 J > 0 \Rightarrow L^1$-local minimality result to the case of rectangular domain with NBC.

- Remark: Morphological instability may only come from wriggling and corrugations.
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- **Spectral analysis:** \( a < \pi \sqrt{\frac{k}{2\gamma}} \) implies that
  \[
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result to the case of rectangular domain with NBC.
Local minimality of 2d lamellar patterns: ingredients

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- **Extension of the Acerbi-Fusco-M.**
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result to the case of **rectangular domain with NBC**

- **Remark**: **Morphological instability** may only come from **wriggling and corrugations**:

\[ \text{corrugation} \quad \text{wriggling} \]
Theorem (M.-Sternberg, 2013)

Fix any $\gamma > 0$. Then for $k(\gamma)$ minimizing the 1d energy

$$J^{1d}(u_k) = k + \frac{\gamma}{12k^2},$$

$u_{k(\gamma)}$ is the unique global minimizer of $J$ in $(0, \varepsilon) \times (0, 1)$ for $\varepsilon$ sufficiently small.
Step 1: $\Gamma$-convergence

Consider (NLIP) posed on a thin rectangle $\Omega_\varepsilon := (0, \varepsilon) \times (0, 1)$:

$$\min J_{\Omega_\varepsilon} (u) := \text{Per}_{\Omega_\varepsilon} (\{ x : u(x) = 1 \}) + \gamma \int_{\Omega_\varepsilon} |\nabla v|^2 \, dx,$$

under the constraint $\int_{\Omega_\varepsilon} u = 0$.

After rescaling onto the (fixed) unit square $\Omega_1$ and dividing by $\varepsilon$, one checks that $J_{\Omega_\varepsilon}$ $\Gamma$-converges as $\varepsilon \to 0$ to the one-dimensional energy

$$J_{1d} (u) := \# \text{ of jumps} + \gamma \int_0^1 v^2 \, dy,$$

for $u \in BV((0,1), \{ \pm 1 \})$, with $\int_0^1 u(y) \, dy = 0$, and where $v = v(y)$ solves

$$-v'' = u \quad \text{for } 0 < y < 1, \quad v'(0) = 0 = v'(1).$$
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$$-v'' = u \quad \text{for } 0 < y < 1, \quad v'(0) = 0 = v'(1).$$
"Perimeter" in 1d just counts number of jumps, located say at \( \{y_j\}_{j=1}^k \). Also criticality condition \( \kappa + 4\gamma v = \text{constant} \) reduces to simply

\[
v(y_j) = \text{constant} \quad (\text{since curvature} = 0)
\]

Easily check that only critical points are \( u_k \) given by

\[
u_k(y) := \begin{cases} 
1 & \text{for } 0 < y < y_1, \ y_2 < y < y_3, \ldots, \ y_{k-1} < y < y_k \\
-1 & \text{for } y_1 < y < y_2, \ y_3 < y < y_4, \ldots, \ y_k < y < 1 
\end{cases}
\]

where \( y_j := \frac{2j-1}{2k} \) for \( j = 1, 2, \ldots, k \) and the derivative of the corresponding \( v_k \) is a "sawtooth function."

(For simplicity only, fix \( m = 0 \).)
Step 2: Explicit calculation of 1d minimizers

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Step 2: Explicit calculation of 1d minimizers

The 5-jump critical point $u_5$ with $m = 0$ and the derivative of the corresponding $v_5$ solving

$$-v^{''}_5 = u_5 \quad \text{for } 0 < y < 1, \quad v'_5(0) = 0 = v'_5(1).$$
Easy to compute that

\[ J_{1d}^{1d}(u_k) = k + \frac{\gamma}{12k^2} \]

so given any \( \gamma \), one can optimize over \( k \) to find global minimizer is \( u_k(\gamma) \), with

\[ k(\gamma) \sim \gamma^{1/3} \text{ for } \gamma \gg 1. \]
Step 2: Explicit calculation of 1d minimizers

Easy to compute that

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\[ k(\gamma) \sim \gamma^{1/3} \text{ for } \gamma \gg 1. \]
Fix $a < \pi \sqrt{\frac{k(\gamma)}{2\gamma}}$. Let $u_\varepsilon$ be a global minimizer in $\Omega_\varepsilon$ and let

$$\tilde{u}_\varepsilon(x, y) := u_\varepsilon\left(\varepsilon \frac{x}{a}, y\right)$$
Step 3: Proof of global minimality

Fix \( a < \pi \sqrt{\frac{k(\gamma)}{2\gamma}} \). Let \( u_\varepsilon \) be a global minimizer in \( \Omega_\varepsilon \) and let

\[
\tilde{u}_\varepsilon(x, y) := u_\varepsilon\left(\frac{\varepsilon}{a} x, y\right)
\]

By \( \Gamma \)-convergence

\[
\tilde{u}_\varepsilon \rightarrow u_{k(\gamma)} \quad \text{in } L^1(\Omega_a)
\]
Fix $a < \pi \sqrt{\frac{k(\gamma)}{2\gamma}}$. Let $u_\varepsilon$ be a global minimizer in $\Omega_\varepsilon$ and let

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By $\Gamma$-convergence

$$\tilde{u}_\varepsilon \rightharpoonup u_{k(\gamma)} \text{ in } L^1(\Omega_a)$$

Let $u_\varepsilon^r$ the function in $BV(\Omega_a)$ obtained by $\frac{a}{\varepsilon} - 1$ reflections of $u_\varepsilon$. 

$$u_\varepsilon^{(1)} = u_\varepsilon$$

$$u_\varepsilon^{(k)}(x, y) := u^{(k-1)}(x, 2(k-1)\varepsilon - y)$$
Step 3: Proof of global minimality: conclusion

Note

\[ \| u_{\varepsilon}^r - u_k(\gamma) \|_{L^1(\Omega_a)} = \| \tilde{u}_{\varepsilon} - u_k(\gamma) \|_{L^1(\Omega_a)} \to 0 \]
Step 3: Proof of global minimality: conclusion

Note

\[ \| u_{\epsilon}^r - u_k(\gamma) \|_{L^1(\Omega_a)} = \| \tilde{u}_{\epsilon} - u_k(\gamma) \|_{L^1(\Omega_a)} \rightarrow 0 \]

Thus for \( \epsilon \) small enough

\[ J_{\Omega_a}(u_{\epsilon}^r) \geq J_{\Omega_a}(u_k(\gamma)) + C \| u_k(\gamma) - u_{\epsilon}^r \|_{L^1(\Omega_a)}^2 \]
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But \( J_{\Omega_a}(u^r_{\varepsilon}) = \frac{a}{\varepsilon} J_{\Omega_{\varepsilon}}(u_{\varepsilon}) \),
Step 3: Proof of global minimality: conclusion

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But \( J_{\Omega_a}(u_\varepsilon^r) = \frac{a}{\varepsilon} J_{\Omega}(u_\varepsilon), \ J_{\Omega_a}(u_k(\gamma)) = \frac{a}{\varepsilon} J_{\Omega}(u_k(\gamma)) \),
Note

\[ \| u_\varepsilon^r - u_{k(\gamma)} \|_{L^1(\Omega_a)} = \| \tilde{u}_\varepsilon - u_{k(\gamma)} \|_{L^1(\Omega_a)} \to 0 \]

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\[ J_{\Omega_a}(u_\varepsilon^r) \geq J_{\Omega_a}(u_{k(\gamma)}) + C \| u_{k(\gamma)} - u_\varepsilon^r \|_{L^1(\Omega_a)}^2 \]

But \( J_{\Omega_a}(u_\varepsilon^r) = \frac{a}{\varepsilon} J_{\Omega_\varepsilon}(u_\varepsilon), \quad J_{\Omega_a}(u_{k(\gamma)}) = \frac{a}{\varepsilon} J_{\Omega_\varepsilon}(u_{k(\gamma)}) \), and

\[ \| u_{k(\gamma)} - u_\varepsilon^r \|_{L^1(\Omega_a)}^2 = \frac{a^2}{\varepsilon^2} \| u_{k(\gamma)} - u_\varepsilon \|_{L^1(\Omega_\varepsilon)}^2 \]
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But \( J_{\Omega_a}(u_\varepsilon^r) = \frac{a}{\varepsilon} J_{\Omega_\varepsilon}(u_\varepsilon), J_{\Omega_a}(u_k(\gamma)) = \frac{a}{\varepsilon} J_{\Omega_\varepsilon}(u_k(\gamma)), \) and

\[ \| u_k(\gamma) - u_\varepsilon^r \|_{L^1(\Omega_a)}^2 = \frac{a^2}{\varepsilon^2} \| u_k(\gamma) - u_\varepsilon \|_{L^1(\Omega_\varepsilon)}^2 \]

Thus, for \( \varepsilon \) small enough

\[ J_{\Omega_\varepsilon}(u_\varepsilon) \geq J_{\Omega_\varepsilon}(u_k(\gamma)) + C \frac{a}{\varepsilon} \| u_k(\gamma) - u_\varepsilon \|_{L^1(\Omega_\varepsilon)}^2 \]
Note

$$\| u^r_{\varepsilon} - u_k(\gamma) \|_{L^1(\Omega_a)} = \| \tilde{u}_{\varepsilon} - u_k(\gamma) \|_{L^1(\Omega_a)} \to 0$$

Thus for $\varepsilon$ small enough

$$J_{\Omega_a}(u^r_{\varepsilon}) \geq J_{\Omega_a}(u_k(\gamma)) + C \| u_k(\gamma) - u^r_{\varepsilon} \|^2_{L^1(\Omega_a)}$$

But $J_{\Omega_a}(u^r_{\varepsilon}) = \frac{a}{\varepsilon} J_{\Omega_\varepsilon}(u_{\varepsilon})$, $J_{\Omega_a}(u_k(\gamma)) = \frac{a}{\varepsilon} J_{\Omega_\varepsilon}(u_k(\gamma))$, and

$$\| u_k(\gamma) - u^r_{\varepsilon} \|^2_{L^1(\Omega_a)} = \frac{a^2}{\varepsilon^2} \| u_k(\gamma) - u_{\varepsilon} \|^2_{L^1(\Omega_\varepsilon)}$$

Thus, for $\varepsilon$ small enough

$$J_{\Omega_\varepsilon}(u_{\varepsilon}) \geq J_{\Omega_\varepsilon}(u_k(\gamma)) + C \frac{a}{\varepsilon} \| u_k(\gamma) - u_{\varepsilon} \|^2_{L^1(\Omega_\varepsilon)}$$

$$\downarrow$$

$$u_{\varepsilon} = u_k(\gamma) \quad \text{for } \varepsilon \text{ small}$$
Thanks for your attention!!!!!!