Thin-film limits of functionals on $\mathcal{A}$-free vector fields and applications

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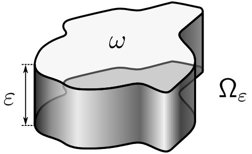
Two thin-film problems in micromagnetics and nonlinear elasticity

Micromagnetics

$\Omega_\varepsilon = \omega \times (0, \varepsilon)$, $\omega \subset \mathbb{R}^2$ bounded.
$m : \Omega_\varepsilon \to \mathbb{S}^2$ magnetization (locally saturated).
$h : \mathbb{R}^3 \to \mathbb{R}^3$ induced magnetic field.

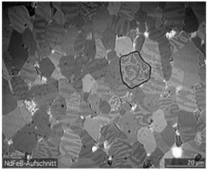


Micromagnetic free energy (*Brown '63, Landau & Lifshitz '35*)

$$\underbrace{\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} \alpha\, |\nabla m|^2 \, \mathrm{d}x}_{\text{exchange energy}} + \underbrace{\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} \varphi(m) \, \mathrm{d}x}_{\text{anisotropy energy}} + \underbrace{\frac{1}{2}\int_{\mathbb{R}^3} |h|^2 \, \mathrm{d}x}_{\text{magnetostatic energy}},$$

if (m, h) meet the static **Maxwell equations**

$$\begin{aligned} \operatorname{div}(m + h) &= 0 \quad \text{in } \mathbb{R}^3 \\ \operatorname{curl} h &= 0 \quad \text{in } \mathbb{R}^3. \end{aligned}$$



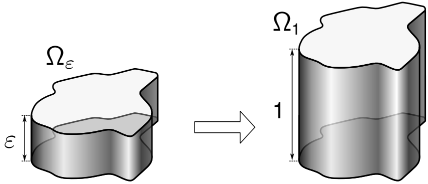
Several grains of NdFeB with magnetic domains made visible via contrast with a Kerr microscope. http://en.wikipedia.org/wiki/Magnetic_domain

Question: Asymptotic behavior as $\varepsilon \downarrow 0$?

Thin ferromagnetic bodies

Thin-film rescaling

$\rightsquigarrow$ work on fix domain Ω_1.



Rescaled energy

$$E_\varepsilon^{\mathbf{mag}}[m,h] = \int_{\Omega_1} \alpha\,|\nabla_\varepsilon m|^2 + \varphi(m)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} |h|^2\,\mathrm{d}x,$$

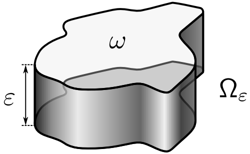
$$\text{if } \underbrace{\begin{pmatrix} \mathrm{div}_\varepsilon(m+h) \\ \mathrm{curl}_\varepsilon\, h \end{pmatrix} = 0 \quad \text{in } \mathbb{R}^3.}_{\text{rescaled Maxwell equations}} \qquad (E_\varepsilon^{\mathbf{mag}} = +\infty, \text{ otherwise.})$$

$\nabla_\varepsilon := (\nabla', 1/\varepsilon\,\partial_3)$
$\nabla' = (\partial_1, \partial_2)$
$\mathrm{div}_\varepsilon := \nabla_\varepsilon \cdot\,,\ \mathrm{curl}_\varepsilon := \nabla_\varepsilon \times$

Our goal: Characterization of the **Γ-limit $E_0^{\mathbf{mag}}$** of $E_\varepsilon^{\mathbf{mag}}$ as $\varepsilon \downarrow 0$, (compare also *Gioia & James '97*).

Classical thin-film limits in nonlinear elasticity

$\Omega_\varepsilon = \omega \times (0, \varepsilon)$, $\omega \subset \mathbb{R}^2$ bounded.
$u : \Omega_\varepsilon \to \mathbb{R}^3$ deformation.



Elastic energy (after thin-film rescaling)

$$I_\varepsilon^{\text{clas}}[u] = \int_{\Omega_1} W(\nabla_\varepsilon u)\,\mathrm{d}x = \int_{\Omega_1} W(\nabla' u | \tfrac{1}{\varepsilon}\partial_3 u)\,\mathrm{d}x.$$

Le Dret & Raoult '95: The limit problem as $\varepsilon \downarrow 0$ is purely **two-dimensional** and characterized by

$$I_0^{\text{clas}}[u] = \int_\omega \mathcal{Q}_2\Big(\min_{b\in\mathbb{R}^3} W\left(\nabla' u(x') \,|\, b\right)\Big)\mathrm{d}x'. \qquad x' := (x_1, x_2)$$

Notice the loss of information on the weak limit of $\frac{1}{\varepsilon}\partial_3 u$.

Bending moment in membrane theory

Incorporating a bending moment (Cosserat vector) b.

Elastic energy with bending moment

$$I_\varepsilon^{\textbf{bend}}[u, b] = \begin{cases} \displaystyle\int_{\Omega_1} W(\nabla' u \,|\, b)\,\mathrm{d}x & \text{if } b = \frac{1}{\varepsilon}\partial_3 u, \\ +\infty & \text{otherwise.} \end{cases}$$

Bouchitté, Fonseca & Mascarenhas '09:
Characterization of the Γ-limit $I_0^{\textbf{bend}}$ of $I_\varepsilon^{\textbf{bend}}$ as $\varepsilon \downarrow 0$.

Open question: Is $I_0^{\textbf{bend}}$ in general a **nonlocal functional**?

Our goal: Give an alternative approach resulting from a more abstract theory.

Dimension reduction in the $\mathcal{A}$-free framework

The $\mathcal{A}$-free framework I

$\Omega \subset \mathbb{R}^d$ bounded Lipschitz domain.
$f : \mathbb{R}^m \to \mathbb{R}$ continuous.

Consider

$$F[u] = \begin{cases} \int_\Omega f(u)\,\mathrm{d}x & \text{if } \boldsymbol{\mathcal{A}u = 0} \text{ in } \Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

We assume that $\boldsymbol{\mathcal{A}}$ is a constant-coefficient first-order linear partial differential operator

$$\boxed{\boldsymbol{\mathcal{A}} := \sum_{k=1}^d A^{(k)} \partial_k} \qquad \text{with } A^{(1)}, \dots, A^{(d)} \in \mathbb{R}^{l \times m},$$

that fulfills **Murat's constant-rank property**, i.e.

$$\boxed{\operatorname{rank} \mathbb{A}(\xi) = \text{const.} \qquad \text{for all } \xi \neq 0.}$$

Examples: $\mathcal{A} = \operatorname{div}$ and $\mathcal{A} = \operatorname{curl}$.

The $\mathcal{A}$-free framework II

Applications:

- Nonlinear elasticity:

  functionals on gradients $\rightsquigarrow$ $\mathcal{A} = \text{curl}$.

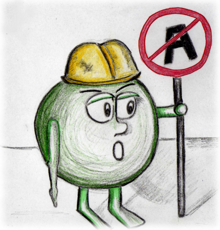
- Micromagnetics:

  static Maxwell equations $\rightsquigarrow$ $\mathcal{A} = \mathcal{A}^{\text{mag}}$, where

$$\mathcal{A}^{\text{mag}} \begin{pmatrix} m \\ h \end{pmatrix} = \begin{pmatrix} \text{div}(m+h) \\ \text{curl}\, h \end{pmatrix}.$$

Literature:

- *Fonseca & Müller '99*
- *Braides, Fonseca & Leoni '00*
- *Fonseca & Krömer '10*
- *Baía, Chermisi, Matias, Santos '11*
- *...*

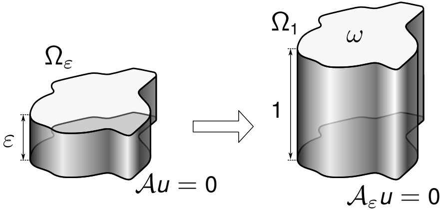


Rescaling and the limit operator $\mathcal{A}_0$

For $\varepsilon > 0$ we consider the rescaled functional $F_\varepsilon : L^p(\Omega_1; \mathbb{R}^m) \to \overline{\mathbb{R}}$,

$$F_\varepsilon[u] = \begin{cases} \int_{\Omega_1} f(u)\,\mathrm{d}x & \text{if } \boldsymbol{\mathcal{A}_\varepsilon u = 0} \text{ in } \Omega_1, \\ +\infty & \text{otherwise,} \end{cases}$$

where $\boldsymbol{\mathcal{A}_\varepsilon} := \sum_{k=1}^{d-1} A^{(k)}\partial_k + \frac{1}{\varepsilon} A^{(d)}\partial_d$.



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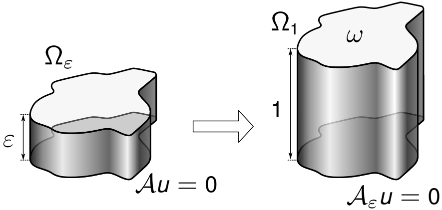


The corresponding limit operator for $\varepsilon \to 0$ is given by

$$\boldsymbol{\mathcal{A}_0} := \left(\begin{cases} [A^{(d)}]^i \partial_d & \text{if } [A^{(d)}]^i \neq 0 \\ \sum_{k=1}^{d-1} [A^{(k)}]^i \partial_k & \text{if } [A^{(d)}]^i = 0 \end{cases} \right)^{i=1,\ldots,l}.$$

Remark: $\mathcal{A}_0$ is in general **not** of constant-rank!

Examples: $\mathcal{A} = \mathrm{div} \rightsquigarrow \mathrm{div}_0\, u = \partial_d u_d$.

$\mathcal{A} = \mathrm{curl} \rightsquigarrow \mathrm{curl}_0\, u = \left(-\partial_3 u_2,\, \partial_3 u_1,\, \partial_1 u_2 - \partial_2 u_1\right)^T$ $(d = 3)$.

Upper and lower bound for the thin-film limit on $\mathcal{A}$-free fields

Theorem 1 [K. & Rindler '14] arXiv:1105.3848

Let $\Omega_1 = \omega \times (0,1) \subset \mathbb{R}^d$ be Lipschitz and f be continuous with standard p-growth and p-coercivity, $p \in (1, +\infty)$. If $\mathcal{A}$ has constant rank such that $\mathcal{A}_0$ satisfies an *extension property*, then:

(*Lower bound*) If $(u_\varepsilon)_\varepsilon \subset \mathcal{U}_{\mathcal{A}_\varepsilon}$ and $u \in L^p(\Omega_1; \mathbb{R}^m)$ with $u_\varepsilon \rightharpoonup u$ in $L^p(\Omega_1; \mathbb{R}^m)$, then $u \in \mathcal{U}_{\mathcal{A}_0}$ and

$$\int_{\Omega_1} \mathcal{Q}^\infty_{\mathcal{A}_0} f(u) \, \mathrm{d}x \leq \liminf_{\varepsilon \to 0} F_\varepsilon[u_\varepsilon].$$

(*Upper bound*) For every $u \in \mathcal{U}_{\mathcal{A}_0}$ there exists a sequence $(u_\varepsilon)_\varepsilon \subset \mathcal{U}_{\mathcal{A}_\varepsilon}$ such that $u_\varepsilon \rightharpoonup u$ in $L^p(\Omega_1; \mathbb{R}^m)$ and

$$\limsup_{\varepsilon \to 0} F_\varepsilon[u_\varepsilon] \leq \int_{\Omega_1} \mathcal{Q}_{\mathcal{A}} f(u) \, \mathrm{d}x.$$

Notice that $\mathcal{U}_{\mathcal{A}_\varepsilon} = \{u \in L^p(\Omega_1; \mathbb{R}^m) : \mathcal{A}_\varepsilon u = 0 \text{ in } \Omega_1\} = L^p(\Omega_1; \mathbb{R}^m) \cap \ker_{\Omega_1} \mathcal{A}_\varepsilon$ and

$$\mathcal{U}_{\mathcal{A}_0} = \{u \in L^p(\Omega_1; \mathbb{R}^m) : \mathcal{A}_0 u = 0 \text{ in } \Omega_1\} = L^p(\Omega_1; \mathbb{R}^m) \cap \ker_{\Omega_1} \mathcal{A}_0.$$

Thin-film Γ-limit in the $\mathcal{A}$-free framework

Recall that

$$F_\varepsilon[u] = \begin{cases} \int_{\Omega_1} f(u)\,\mathrm{d}x & \text{if } \boldsymbol{\mathcal{A}_\varepsilon u = 0} \text{ in } \Omega_1, \\ +\infty & \text{otherwise,} \end{cases} \qquad u \in L^p(\Omega_1;\mathbb{R}^m).$$

Corollary 2 [K. & Rindler '14] arXiv:1105.3848

Let the assumptions of Theorem 1 hold. If additionally,

$$\mathcal{Q}_{\mathcal{A}} f \leq \mathcal{Q}^\infty_{\mathcal{A}_0} f \quad \text{(e.g., if } f \text{ is asymptotically } \mathcal{A}_0\text{-quasiconvex),}$$

then $\boldsymbol{F_\varepsilon} \xrightarrow{\Gamma} \boldsymbol{F_0}$ with respect to weak convergence in $L^p(\Omega_1;\mathbb{R}^m)$ as $\varepsilon \downarrow 0$.

Here,

$$F_0[u] := \begin{cases} \int_{\Omega_1} \boldsymbol{\mathcal{Q}_{\mathcal{A}} f}(u)\,\mathrm{d}x & \text{if } \boldsymbol{\mathcal{A}_0 u = 0} \text{ in } \Omega_1, \\ +\infty & \text{otherwise,} \end{cases} \qquad u \in L^p(\Omega_1;\mathbb{R}^m).$$

Proof of Theorem 1: Basic ideas

Key tool: Projection onto $\mathcal{A}_\varepsilon$-free vector fields with uniformly bounded constants (constant-rank property) (*Fonseca & Müller '99, K. & Rindler '14*).

Lower bound: Young measure approach (*Fonseca & Müller '99*)
Ingredients: localization by blow-up, **decomposition lemma**

Upper bound: Explicit construction of a recovery sequence
Ingredients: relaxation theory in $\mathcal{A}$-framework (*Braides, Fonseca & Leoni '00*)
+ **Proposition 5** + diagonalization argument

Proposition 5
For $u \in \mathcal{U}_{\mathcal{A}_0}$ there exists $(u_\varepsilon)_\varepsilon \subset \mathcal{U}_{\mathcal{A}_\varepsilon}$ such that $u_\varepsilon \to u$ in $L^p(\Omega_1;\mathbb{R}^m)$.

Lemma 6 (Convergence of 'symbols')
It holds that $\mathbb{A}_\varepsilon \longrightarrow \tilde{\mathbb{A}}_{\mathbf{0}}$, i.e. $\mathbb{P}_{\mathcal{A}_\varepsilon}(\xi) \to \tilde{\mathbb{P}}_0(\xi)$ for $\xi \in \mathbb{R}^d$.

⚠ Notice that $\tilde{\mathbb{A}}_0(\xi) \neq \mathbb{A}_0(\xi)$ if $\xi_d = 0$.

$\rightsquigarrow$ explicit construction on plane waves in $\mathbb{R}^{d-1} \times \{0\}$.

Proof of Theorem 1: Discussion

Fourier methods...

- ... allow for intuitive, algebraic arguments in Fourier space.
- ... give insight into the structure of the dimension reduction problem.

Price to pay:

Extension property of $\mathcal{A}_0$-free fields
in order to work on general domains Ω_1.

Assumption (Approximate extension property)
Let $\omega \subset\subset Q^{d-1}$. For every $u \in \mathcal{U}_{\mathcal{A}_0}$ there exists $(u_j)_j \subset L^p(Q^d;\mathbb{R}^m) \cap \ker_{\mathbb{T}^d} \mathcal{A}_0$ such that $u_j \to u$ in $L^p(\Omega_1;\mathbb{R}^m)$.

Examples: div_0 and curl_0 satisfy the extension property.

Back to the applications

Elastic thin-films in the curl-free formulation

Recall the classical elastic energy:

$$I_\varepsilon^{\mathbf{clas}}[u] = \int_{\Omega_1} W(\nabla_\varepsilon u)\,\mathrm{d}x, \qquad u \in W^{1,p}(\Omega_1;\mathbb{R}^3).$$

Within the curl-free framework we define

$$\boldsymbol{I_\varepsilon}[H] := \begin{cases} \displaystyle\int_{\Omega_1} W(H)\,\mathrm{d}x & \text{if } \mathbf{curl}_{\boldsymbol{\varepsilon}}\, \boldsymbol{H} = \mathbf{0}, \\ +\infty & \text{otherwise,} \end{cases} \qquad H \in L^p(\Omega_1;\mathbb{R}^{3\times 3}).$$

Equivalence between the energies I_ε and $I_\varepsilon^{\mathbf{bend}}$:

$$\boldsymbol{I_\varepsilon^{\mathbf{bend}}}[u,b] = \boldsymbol{I_\varepsilon}[H] \qquad \Leftrightarrow \qquad H = (\nabla' u \,|\, b),$$

where $$I_\varepsilon^{\mathbf{bend}}[u,b] = \begin{cases} \displaystyle\int_{\Omega_1} W(\nabla' u \,|\, b)\,\mathrm{d}x & \text{if } b = \frac{1}{\varepsilon}\partial_3 u, \\ +\infty & \text{otherwise,} \end{cases} \qquad \begin{array}{c} u \in W^{1,p}(\Omega_1;\mathbb{R}^3) \\ b \in L^p(\Omega_1,\mathbb{R}^3) \end{array}.$$

Bending of elastic thin-films

Recall the elastic energy with bending moment

$$I_\varepsilon^{\textbf{bend}}[u,b] = \begin{cases} \int_{\Omega_1} W(\nabla' u \,|\, b)\,\mathrm{d}x & \text{if } b = \frac{1}{\varepsilon}\partial_3 u, \\ +\infty & \text{otherwise,} \end{cases} \qquad \begin{array}{c} u \in W^{1,p}(\Omega_1;\mathbb{R}^3) \\ b \in L^p(\Omega_1;\mathbb{R}^3) \end{array} .$$

Corollary 7

If W is asymptotically curl_0-quasiconvex, the Γ-limit of $I_\varepsilon^{\textbf{bend}}$ with respect to weak convergence in $W^{1,p}(\Omega_1;\mathbb{R}^3) \times L^p(\Omega_1;\mathbb{R}^3)$ is

$$I_0^{\textbf{bend}}[u,b] = \int_{\Omega_1} W\big(\nabla' u(x') \,|\, b(x)\big)\,\mathrm{d}x.$$

Proof: Apply Corollary 2 with $F_\varepsilon = I_\varepsilon$, $f = W$ and $\mathcal{A} = \mathrm{curl}$ and use

$$\mathrm{curl}_0\, H = 0 \quad \Leftrightarrow \quad H(x) = \big(\nabla' u(x') | b(x)\big).$$

Remark: This is in agreement with the result of *Bouchitté, Fonseca & Mascarenhas '09*.

Thin-film limit in micromagnetics

Theorem 8 [K. '11] Quart. Appl. Math. (2013)

The Γ-limit of $E_\varepsilon^{\textbf{mag}}$ for $\varepsilon \downarrow 0$ regarding weak convergence in $W^{1,2}(\Omega_1;\mathbb{R}^3) \times L^2(\mathbb{R}^3;\mathbb{R}^3)$ exists and is finite with

$$E_0^{\textbf{mag}}[m,h] = \int_\omega \alpha|\nabla' m|^2 + \underbrace{\varphi(m) + \frac{1}{2}m_3^2}_{\text{effective anisotropy energy}} \, \mathrm{d}x',$$

if $m \in W^{1,2}(\omega;\mathbb{S}^2)$ and $h = -(0,0,\,m_3) \in L^2(\mathbb{R}^2;\mathbb{R}^3)$.

Interpretation:

- $E_0^{\textbf{mag}}$ is purely two-dimensional (independence of x_3) .
- $E_0^{\textbf{mag}}$ is local (Maxwell equations disappeared).
- Magnetizations pointing out of plane are unfavorable.
- Minimizers of $E_0^{\textbf{mag}}$ are uniformly magnetized (single domain).

Remark: $E_0^{\textbf{mag}}$ coincides with limit energy of *Gioia & James '97*.

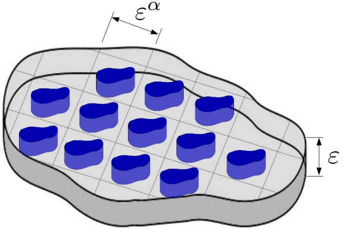
Outlook

The question of locality

Open problem: Is the Γ-limit F_0 of F_ε in general a local functional?

Conjecture *(Bouchitté, Fonseca & Mascarenhas '09)*: The functional $I_0^{\mathbf{bend}}$ may be nonlocal.

Heterogeneous thin films



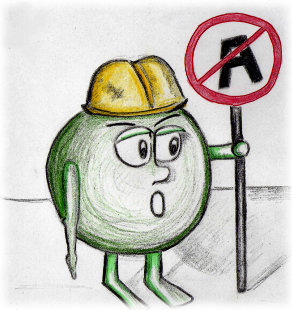
For $\alpha > 0$ we define

$$F_\varepsilon^\alpha[u] = \begin{cases} \int_{\Omega_1} f\left(\frac{x'}{\varepsilon^\alpha}, u(x)\right) \mathrm{d}x & u \in \mathcal{U}_{\mathcal{A}_\varepsilon} = L^p(\Omega_1; \mathbb{R}^m) \cap \ker_{\Omega_1} \mathcal{A}_\varepsilon, \\ +\infty & \text{otherwise.} \end{cases}$$

Proposition 9 [K. & Krömer '14]

Let $\Omega_1 = (0,1)^d$, $p > 1$, $\mathcal{A}$ be a constant-rank operator, and $\alpha < 1$.
Then there exists an integrand f that is convex in the second variable such that $F_0^\alpha := \Gamma\text{-}\lim F_\varepsilon^\alpha$ is nonlocal.

Thank you!



$\mathcal{A}$-quasiconvexity and asymptotic $\mathcal{A}_0$-quasiconvexity

$\mathbb{T}^d$ is the d-dimensional torus, which results from glueing opposite edges of $Q^d := (0,1)^d$.

The **$\mathcal{A}$-quasiconvex envelope $\mathcal{Q}_{\mathcal{A}} f$** is defined as

$$\mathcal{Q}_{\mathcal{A}} f(v) = \inf\Big\{ \int_{Q^d} f(v + w(x))\,\mathrm{d}x : w \in C^\infty(\mathbb{T}^d;\mathbb{R}^m), \mathcal{A}w = 0 \text{ in } \mathbb{T}^d, \int_{Q^d} w\,\mathrm{d}x = 0 \Big\}.$$

Definition: A function $f:\mathbb{R}^m \to \mathbb{R}$ is **$\mathcal{A}$-quasiconvex**, if

$$f(v) \leq \int_{Q^d} f\big(v + w(x)\big)\,\mathrm{d}x$$

for all $v \in \mathbb{R}^m$ and all $w \in C^\infty(\mathbb{T}^d,\mathbb{R}^m)$ with $\mathcal{A}w = 0$ in $\mathbb{T}^d$ and $\int_{Q^d} w\,\mathrm{d}x = 0$.

Examples:
$\mathcal{A} = \mathrm{div}$, then $\mathcal{A}$-quasiconvexity is convexity $\rightsquigarrow$ $\mathcal{Q}_{\mathrm{div}} f = f^{**} = \mathcal{Q}_{\mathrm{div}_0} f = \mathcal{Q}^\infty_{\mathrm{div}_0} f$.
$\mathcal{A} = \mathrm{curl}$, then $\mathcal{A}$-quasiconvexity is quasiconvexity $\rightsquigarrow$ $\mathcal{Q}_{\mathrm{curl}} f = \mathcal{Q} f$, $\mathcal{Q}^\infty_{\mathrm{curl}_0} f \leq \mathcal{Q}_{\mathrm{curl}_0} f$.

$\mathcal{A}$-quasiconvexity and asymptotic $\mathcal{A}_0$-quasiconvexity

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Further, we define $\mathcal{Q}^\infty_{\mathcal{A}_0} f(v) = \lim_{\eta\to\infty} \mathcal{Q}^\eta_{\mathcal{A}_0} f(v)$ for $v \in \mathbb{R}^m$ with

$$\mathcal{Q}^\eta_{\mathcal{A}_0} f(v) = \inf\Big\{\int_{Q^d} f(v + w(x))\,\mathrm{d}x : w \in C^\infty(\mathbb{T}^d;\mathbb{R}^m),$$
$$\eta\|\mathcal{A}_0 w\|_{W^{-1,1}(\mathbb{T}^d;\mathbb{R}^l)} \leq 1, \int_{Q^d} w\,\mathrm{d}x = 0\Big\},$$

and say that f is asymptotically $\mathcal{A}_0$-quasiconvex if $\mathcal{Q}^\infty_{\mathcal{A}_0} f = f$.

Examples:
$\mathcal{A} = \mathrm{div}$, then $\mathcal{A}$-quasiconvexity is convexity $\rightsquigarrow$ $\mathcal{Q}_{\mathrm{div}} f = f^{**} = \mathcal{Q}_{\mathrm{div}_0} f = \mathcal{Q}^\infty_{\mathrm{div}_0} f$.
$\mathcal{A} = \mathrm{curl}$, then $\mathcal{A}$-quasiconvexity is quasiconvexity $\rightsquigarrow$ $\mathcal{Q}_{\mathrm{curl}} f = \mathcal{Q} f$, $\mathcal{Q}^\infty_{\mathrm{curl}_0} f \leq \mathcal{Q}_{\mathrm{curl}_0} f$.

Key tool: Projection onto $\mathcal{A}$-free fields

Lemma 3 [Fonseca & Müller '99]
Let $p \in (1, \infty)$ and $\mathcal{A}$ a constant-rank operator. Then there exists a bounded projection operator $\mathcal{P}_{\mathcal{A}} : L^p(\mathbb{T}^d; \mathbb{R}^m) \to L^p(\mathbb{T}^d; \mathbb{R}^m)$ such that for all $u \in L^p(\mathbb{T}^d; \mathbb{R}^m)$:

- $\mathcal{P}_{\mathcal{A}} u$ is $\mathcal{A}$-free in $\mathbb{T}^d$,
- $\|\mathcal{P}_{\mathcal{A}} u\|_{L^p(\mathbb{T}^d;\mathbb{R}^m)} \leq C \|u\|_{L^p(\mathbb{T}^d;\mathbb{R}^m)}$,
- $\|\mathcal{P}_{\mathcal{A}} u - u\|_{L^p(\mathbb{T}^d;\mathbb{R}^m)} \leq C \|\mathcal{A}u\|_{W^{-1,p}(\mathbb{T}^d;\mathbb{R}^l)}$,

with a constant $C > 0$.

Proof: Based on Fourier methods. Define

$$\mathcal{P}_{\mathcal{A}}(u)(x) = \hat{u}(0) + \sum_{\xi \in \mathbb{Z}^d \setminus \{0\}} \mathbb{P}_{\mathcal{A}}(\xi)\, \hat{u}(\xi)\, e^{2\pi i x \cdot \xi},$$

where $\mathbb{P}_{\mathcal{A}}(\xi)$ is the orthogonal projection onto $\ker \mathbb{A}(\xi)$ in Fourier space.

Important ingredients: **constant-rank property**

Key tool: Projection onto $\mathcal{A}_\varepsilon$-free fields

Lemma 3 [Fonseca & Müller '99, K. & Rindler '14]
Let $p \in (1, \infty)$ and $\mathcal{A}$ a constant-rank operator. Then there exists a family of bounded projection operators $\mathcal{P}_{\mathcal{A}_\varepsilon} : L^p(\mathbb{T}^d; \mathbb{R}^m) \to L^p(\mathbb{T}^d; \mathbb{R}^m)$ such that for all $u \in L^p(\mathbb{T}^d; \mathbb{R}^m)$:

- $\mathcal{P}_{\mathcal{A}_\varepsilon} u$ is $\mathcal{A}_\varepsilon$-free in $\mathbb{T}^d$,
- $\|\mathcal{P}_{\mathcal{A}_\varepsilon} u\|_{L^p(\mathbb{T}^d;\mathbb{R}^m)} \leq C \|u\|_{L^p(\mathbb{T}^d;\mathbb{R}^m)}$,
- $\|\mathcal{P}_{\mathcal{A}_\varepsilon} u - u\|_{L^p(\mathbb{T}^d;\mathbb{R}^m)} \leq C \|\mathcal{A}_\varepsilon u\|_{W^{-1,p}(\mathbb{T}^d;\mathbb{R}^l)}$,

with a uniform constant $C > 0$.

Proof: Based on Fourier methods. Define

$$\mathcal{P}_{\mathcal{A}_\varepsilon}(u)(x) = \hat{u}(0) + \sum_{\xi \in \mathbb{Z}^d \setminus \{0\}} \mathbb{P}_{\mathcal{A}_\varepsilon}(\xi)\, \hat{u}(\xi)\, \mathrm{e}^{2\pi \mathrm{i} x \cdot \xi},$$

where $\mathbb{P}_{\mathcal{A}_\varepsilon}(\xi)$ is the orthogonal projection onto $\ker \mathbb{A}_\varepsilon(\xi)$ in Fourier space.

Important ingredients: **constant-rank property**

+ scaling argument for Fourier multipliers on $\mathbb{R}^d$.

Necessity of constant-rank property for projection results

Counterexample [Krömer '12]:

Let $d = m = 2$, $l = 1$ and consider the operator $\mathrm{div}_0 : L^2(\mathbb{T}^2;\mathbb{R}^2) \to W^{-1,2}(\mathbb{T}^2)$,

$$\mathrm{div}_0\, u = \partial_2 u_2.$$

The orthogonal projection onto div_0-free fields is given by

$$\mathcal{P}_{\mathrm{div}_0} u(x) = u_1(x)e_1 + \left(\int_0^1 u_2(x_1, s)\,\mathrm{d}s\right)e_2.$$

For the sequence $(u_j)_j$ defined by

$$u_j(x) = \sin(2\pi j x_1)\sin(2\pi x_2)e_2, \qquad x \in Q^2,$$

one finds that

- $\|\mathrm{div}_0\, u_j\|_{W^{-1,2}(\mathbb{T}^2)} \to 0$ for $j \to \infty$,

$$\text{since } \mathrm{div}_0\, u_j = 2\pi \sin(2\pi j x_1)\cos(2\pi x_2) \rightharpoonup 0 \text{ in } L^2(\mathbb{T}^2).$$

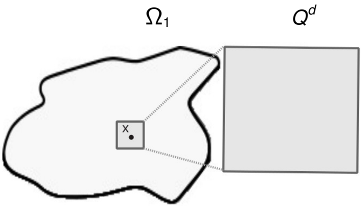
But,
- $\|\mathcal{P}_{\mathrm{div}_0} u_j - u_j\|_{L^2(\mathbb{T}^2;\mathbb{R}^2)} = \|u_j\|_{L^2(\mathbb{T}^2;\mathbb{R}^2)} = \int_0^1 \sin^2(2\pi s)\,\mathrm{d}s = \frac{1}{2} > 0.$

Proof of Theorem 1: Lower bound

Method: Young measure approach (*Fonseca & Müller '99*)

Localization by blow-up

- $(u_\varepsilon)_\varepsilon \subset L^p(\Omega_1;\mathbb{R}^m)$
- $u_\varepsilon \rightharpoonup u$ in $L^p(\Omega_1;\mathbb{R}^m)$
- $\mathcal{A}_\varepsilon u_\varepsilon = 0$ in Ω_1
- $u_\varepsilon \xrightarrow{YM} (\nu_x)_{x\in\Omega_1}$



- $(v_\varepsilon^x)_\varepsilon \subset L^p(Q^d;\mathbb{R}^m)$ bounded
- $\int_{Q^d} v_\varepsilon^x \,\mathrm{d}y = u(x)$
- $\mathcal{A}_\varepsilon v_\varepsilon^x = 0$ in Q^d
- $v_\varepsilon^x \xrightarrow{YM} (\nu_x)_{y\in Q^d}$

$$\liminf_{\varepsilon\to 0} \int_{\Omega_1} f(u_\varepsilon)\,\mathrm{d}x \geq \int_{\Omega_1} \langle f, \nu_x\rangle \,\mathrm{d}x$$

$$\langle f, \nu_x\rangle \stackrel{!}{=} \lim_{\varepsilon\to 0} \int_{Q^d} f(v_\varepsilon^x)\,\mathrm{d}y$$

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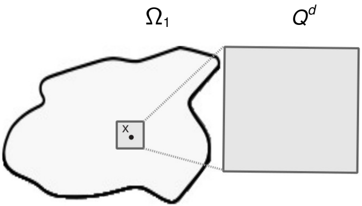

- $(z^x_\varepsilon)_\varepsilon \subset L^p(Q^d;\mathbb{R}^m)$ p-equiintegrable
- $\int_{Q^d} z^x_\varepsilon \,\mathrm{d}y = u(x)$
- $\mathcal{A}_0 z^x_\varepsilon \to 0$ in $W^{-1,1}(\mathbb{T}^d;\mathbb{R}^m)$
- $z^x_\varepsilon \xrightarrow{YM} (\nu_x)_{y\in Q^d}$

$$\liminf_{\varepsilon\to 0} \int_{\Omega_1} f(u_\varepsilon)\,\mathrm{d}x \geq \int_{\Omega_1} \langle f,\nu_x\rangle\,\mathrm{d}x$$

$$\langle f,\nu_x\rangle = \lim_{\varepsilon\to 0}\int_{Q^d} f(z^x_\varepsilon)\,\mathrm{d}y \geq \mathcal{Q}^\infty_{\mathcal{A}_0} f(u(x))$$

Lemma 4 (Local decomposition lemma)

If $(v_\varepsilon)_\varepsilon \subset L^p(Q^d;\mathbb{R}^m)$ with $v_\varepsilon \rightharpoonup v$ in $L^p(Q^d;\mathbb{R}^m)$ and $\mathcal{A}_\varepsilon v_\varepsilon = 0$ in Q^d, then there is a p-equiintegrable sequence $(z_\varepsilon)_\varepsilon \subset L^p(Q^d;\mathbb{R}^m)$ such that $z_\varepsilon - v_\varepsilon \to 0$ in $L^1(Q^d;\mathbb{R}^m)$,

$$\int_{Q^d} v_\varepsilon\,\mathrm{d}y = \int_{Q^d} z_\varepsilon\,\mathrm{d}y \quad \text{and} \quad \mathcal{A}_0 z_\varepsilon \to 0 \text{ in } W^{-1,1}(\mathbb{T}^d;\mathbb{R}^m).$$

Proof of Theorem 1: Upper bound

Method: Explicit construction of a recovery sequence

Ingredients: relaxation theory in $\mathcal{A}$-framework (*Braides, Fonseca & Leoni '00*)

+ **Proposition 5** + diagonalization argument

Proposition 5

For $u \in \mathcal{U}_{\mathcal{A}_0}$ there exists $(u_\varepsilon)_\varepsilon \subset \mathcal{U}_{\mathcal{A}_\varepsilon}$ such that $u_\varepsilon \to u$ in $L^p(\Omega_1;\mathbb{R}^m)$.

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Lemma 6 (Convergence of 'symbols')

It holds that $\mathbb{A}_\varepsilon \longrightarrow \tilde{\mathbb{A}}_{\mathbf{0}}$, i.e. $\mathbb{P}_{\mathcal{A}_\varepsilon}(\xi) \to \tilde{\mathbb{P}}_{\mathcal{A}_0}(\xi)$ for $\xi \in \mathbb{R}^d$.

⚠ Notice that $\tilde{\mathbb{A}}_0(\xi) \neq \mathbb{A}_0(\xi)$ if $\xi_d = 0$.

If $\xi_d \neq 0$, then $r = \operatorname{rank} \mathbb{A}_\varepsilon(\xi) = \operatorname{rank} \mathbb{A}_0(\xi)$ by the constant-rank property.

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We split $u = u^{(1)} + u^{(2)}$:

- $\hat{u}^{(1)}(\xi) = \begin{cases} \hat{u}(\xi) & \text{if } \xi_d \neq 0 \\ 0 & \text{if } \xi_d = 0 \end{cases}$
- $\hat{u}^{(2)}(\xi) = \begin{cases} 0 & \text{if } \xi_d \neq 0 \\ \hat{u}(\xi) & \text{if } \xi_d = 0 \end{cases}$
- $\tilde{\mathbb{A}}_0(\xi)\hat{u}^{(1)}(\xi) = 0$ for all $\xi \in \mathbb{R}^d$
- $u^{(2)}(x) = u^{(2)}(x')$ for $x \in Q^d$
- $u_\varepsilon^{(1)} := \mathcal{P}_{\mathcal{A}_\varepsilon} u^{(1)}$
- $u_\varepsilon^{(2)}(x) := u^{(2)}(x') - \varepsilon x_d (A^{(d)})^\dagger \mathcal{A}' u^{(2)}(x')$

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Proof of Theorem 3

Recall that

$$E_\varepsilon^{\mathbf{mag}}[m,h] = \int_{\Omega_1} \alpha\, |\nabla_\varepsilon m|^2 + \varphi(m)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} |h|^2\,\mathrm{d}x,$$

$$\text{if } (m,h) \in \ker_{\mathbb{R}^3} \mathcal{A}_\varepsilon^{\mathbf{mag}} \text{ with } |m| = 1 \quad \big(E_\varepsilon^{\mathbf{mag}} = +\infty, \text{ otherwise}\big),$$

and

$$F_\varepsilon[u] = \begin{cases} \int_{\Omega_1} f(u)\,\mathrm{d}x & \text{if } u \in \mathcal{U}_{\mathcal{A}_\varepsilon} = \ker_{\Omega_1} \mathcal{A}_\varepsilon, \\ \infty & \text{otherwise.} \end{cases}$$

Specific characteristics of $E_\varepsilon^{\mathbf{mag}}$ (in comparison to F_ε):

- $\Omega_1 = \mathbb{R}^3$ ✓ $\rightsquigarrow$ replace Fourier series by Fourier transforms
- non-convex constraint
- integrand with derivatives of m

Proof of Theorem 3: Upper bound

Let $(m, h) \in \ker_{\mathbb{R}^3} \mathcal{A}_0^{\textbf{mag}}$ with $|m| = 1$ and $\partial_3 m = 0$ in Ω_1.

In analogy to Proposition 5 there exists a sequence $(\tilde{m}_j, \tilde{h}_j) \subset \ker_{\mathbb{R}^3} \mathcal{A}_{\varepsilon_j}^{\mathrm{mag}}$ with

$$(\tilde{m}_j, \tilde{h}_j) \to (m, h) \qquad \text{in } L^2(\mathbb{R}^3; \mathbb{R}^3) \times L^2(\mathbb{R}^3; \mathbb{R}^3) \text{ as } j \to 0.$$

Problem: Regularity and non-convex constraint **NOT satisfied**!

Set

$$\boxed{\begin{aligned} \boldsymbol{m_j} &= m \\ \boldsymbol{h_j} &= \mathcal{P}_{\mathrm{curl}_{\varepsilon_j}}\big(\tilde{h}_j - m + \tilde{m}_j\big) \end{aligned}} \qquad \text{for all } j \in \mathbb{N}.$$

Idea: Keep magnetization fixed and adjust the induced field!

The sequence $(m_j, h_j)_j$ is a recovery sequence.