Local and global minimality results for a nonlocal isoperimetric problem on $\mathbb{R}^N$

Riccardo Cristoferi - SISSA
joint work with Marco Bonacini - University of Heidelberg

Trends in Non-Linear Analysis
Lisbon

July 31th, 2014

The starting point

The starting point:

F. Cagnetti, M.G. Mora, M. Morini,
A second order minimality condition for the Mumford-Shah functional.
Calc. Var. Partial Differential Equations **33** (2008), 37-74

and the development of the technique:

N. Fusco, M. Morini,
Equilibrium configurations of epitaxially strained elastic films: second order minimality conditions and qualitative properties of the solutions.
Arch. Rat. Mech. Anal. **203** (2012), 247-327.

The problem

Let $E \subset \mathbb{R}^N$, where $N \geq 2$, and let $\alpha \in (0, N)$.
Define the functional

The problem

Let $E \subset \mathbb{R}^N$, where $N \geq 2$, and let $\alpha \in (0, N)$.
Define the functional

$$\mathcal{F}(E) := \mathcal{P}(E)$$

The problem

Let $E \subset \mathbb{R}^N$, where $N \geq 2$, and let $\alpha \in (0, N)$.
Define the functional

$$\mathcal{F}(E) := \mathcal{P}(E) + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\chi_E(x)\chi_E(y)}{|x-y|^\alpha} \, \mathrm{d}x\mathrm{d}y$$

The problem

Let $E \subset \mathbb{R}^N$, where $N \geq 2$, and let $\alpha \in (0, N)$.

We are interested in the behaviour of the following volume constraint minimization problem

$$\min \left\{ \mathcal{F}(E) := \mathcal{P}(E) + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\chi_E(x)\chi_E(y)}{|x-y|^\alpha} \, \mathrm{d}x\mathrm{d}y \ : |E| = m \right\} ,$$

with respect to the variation of the parameters α and $m > 0$.

Heuristic

Let $E \subset \mathbb{R}^N$ with $|E| = m$.
Define

$$\widetilde{E} := \left(\frac{|B_1|}{m} \right)^{\frac{1}{N}} E \,.$$

Then

$$\mathcal{F}(E) = \mathcal{P}(E) + \mathcal{NL}_\alpha(E) = \left(\frac{m}{|B_1|} \right)^{\frac{N-1}{N}} \left[\mathcal{P}(\widetilde{E}) + \left(\frac{m}{|B_1|} \right)^{\frac{N+1-\alpha}{N}} \mathcal{NL}_\alpha(\widetilde{E}) \right] .$$

Heuristic

Let $E \subset \mathbb{R}^N$ with $|E| = m$.
Define

$$\widetilde{E} := \left(\frac{|B_1|}{m} \right)^{\frac{1}{N}} E \,.$$

Then

$$\mathcal{F}(E) = \mathcal{P}(E) + \mathcal{NL}_\alpha(E) = \left(\frac{m}{|B_1|} \right)^{\frac{N-1}{N}} \left[\mathcal{P}(\widetilde{E}) + \left(\frac{m}{|B_1|} \right)^{\frac{N+1-\alpha}{N}} \mathcal{NL}_\alpha(\widetilde{E}) \right] .$$

Hence consider the volume constraint minimization problem:

$$\min \left\{ \mathcal{F}_{\alpha,\gamma}(E) := \mathcal{P}(E) + \gamma \mathcal{NL}_\alpha(E) \ : |E| = |B_1| \right\} \,.$$

Heuristic

$$\mathcal{F}_{\alpha,\gamma}(E) = \mathcal{P}(E) + \gamma \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\chi_E(x)\chi_E(y)}{|x-y|^\alpha} \, \mathrm{d}x\mathrm{d}y \, .$$

Heuristic

$$\mathcal{F}_{\alpha,\gamma}(E) = \mathcal{P}(E) + \gamma \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\chi_E(x)\chi_E(y)}{|x-y|^\alpha} \,\mathrm{d}x\mathrm{d}y\,.$$

Notice that (with volume constraint)

- $\mathcal{P}$ is **minimized** by the ball (by the classical Isoperimetric Problem)
- $\mathcal{NL}_\alpha$ is **maximized** by the ball (by Riesz rearrangemenet)

Heuristic

$$\mathcal{F}_{\alpha,\gamma}(E) = \mathcal{P}(E) + \gamma \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\chi_E(x)\chi_E(y)}{|x-y|^\alpha} \, \mathrm{d}x\mathrm{d}y \,.$$

Previous results:

- for $N = 2$:

Reference:
H. Knüpfer, C.B. Muratov,
On an isoperimetric problem with a competing non-local term. I. The planar case.
Comm. Pure Appl. Math. **66** (2013), 1129–1162.

Heuristic

$$\mathcal{F}_{\alpha,\gamma}(E) = \mathcal{P}(E) + \gamma \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\chi_E(x)\chi_E(y)}{|x-y|^\alpha} \, \mathrm{d}x\mathrm{d}y \, .$$

Previous results:

- for $N = 2$:
  - existence of minimizers for γ small

Reference:
H. Knüpfer, C.B. Muratov,
On an isoperimetric problem with a competing non-local term. I. The planar case.
Comm. Pure Appl. Math. **66** (2013), 1129–1162.

Heuristic

$$\mathcal{F}_{\alpha,\gamma}(E) = \mathcal{P}(E) + \gamma \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\chi_E(x)\chi_E(y)}{|x-y|^\alpha} \,\mathrm{d}x\mathrm{d}y\,.$$

Previous results:

- for $N = 2$:
  - existence of minimizers for γ small
  - no minimizers for γ big

Reference:
H. Knüpfer, C.B. Muratov,
On an isoperimetric problem with a competing non-local term. I. The planar case.
Comm. Pure Appl. Math. **66** (2013), 1129–1162.

Heuristic

$$\mathcal{F}_{\alpha,\gamma}(E) = \mathcal{P}(E) + \gamma \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\chi_E(x)\chi_E(y)}{|x-y|^\alpha} \,\mathrm{d}x\mathrm{d}y \,.$$

Previous results:

- for $N = 2$:
  - existence of minimizers for γ small
  - no minimizers for γ big
  - the ball is the unique global minimizer for γ small

Reference:
H. Knüpfer, C.B. Muratov,
On an isoperimetric problem with a competing non-local term. I. The planar case.
Comm. Pure Appl. Math. **66** (2013), 1129–1162.

Heuristic

$$\mathcal{F}_{\alpha,\gamma}(E) = \mathcal{P}(E) + \gamma \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\chi_E(x)\chi_E(y)}{|x-y|^\alpha} \, \mathrm{d}x\mathrm{d}y \, .$$

Previous results:

- for $N = 2$:
  - existence of minimizers for γ small
  - no minimizers for γ big
  - the ball is the unique global minimizer for γ small
- for $N \geq 3$ existence of minimizers for γ small

Reference:
H. Knüpfer, C.B. Muratov,
On an isoperimetric problem with a competing non-local term. II. The general case.
Comm. Pure Appl. Math. (accepted paper).

Heuristic

$$\mathcal{F}_{\alpha,\gamma}(E) = \mathcal{P}(E) + \gamma \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\chi_E(x)\chi_E(y)}{|x-y|^\alpha} \, \mathrm{d}x\mathrm{d}y \, .$$

Previous results:

- for $N = 2$:
  - existence of minimizers for γ small
  - no minimizers for γ big
  - the ball is the unique global minimizer for γ small
- for $N \geq 3$ existence of minimizers for γ small
- for $N \geq 3$ and $\alpha \in (0, 2)$ non existence of minimizers for γ big

Reference:
H. Knüpfer, C.B. Muratov,
On an isoperimetric problem with a competing non-local term. II. The general case.
Comm. Pure Appl. Math. (accepted paper).

Heuristic

$$\mathcal{F}_{\alpha,\gamma}(E) = \mathcal{P}(E) + \gamma \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\chi_E(x)\chi_E(y)}{|x-y|^\alpha} \, \mathrm{d}x\mathrm{d}y \, .$$

Previous results:

- for $N = 2$:
  - existence of minimizers for γ small
  - no minimizers for γ big
  - the ball is the unique global minimizer for γ small
- for $N \geq 3$ existence of minimizers for γ small
- for $N \geq 3$ and $\alpha \in (0,2)$ non existence of minimizers for γ big
- for $3 \leq N \leq 7$ the ball is the unique global minimizer for γ small

Reference:
H. Knüpfer, C.B. Muratov,
On an isoperimetric problem with a competing non-local term. II. The general case.
Comm. Pure Appl. Math. (accepted paper).

Heuristic

$$\mathcal{F}_{\alpha,\gamma}(E) = \mathcal{P}(E) + \gamma \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\chi_E(x)\chi_E(y)}{|x-y|^\alpha}\,\mathrm{d}x\mathrm{d}y\,.$$

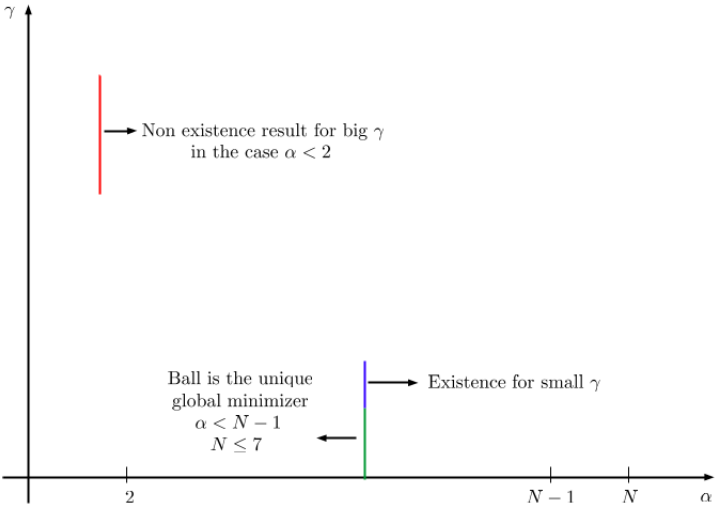
Previous results:

- for $N = 2$:
  - existence of minimizers for γ small
  - no minimizers for γ big
  - the ball is the unique global minimizer for γ small
- for $N \geq 3$ existence of minimizers for γ small
- for $N \geq 3$ and $\alpha \in (0,2)$ non existence of minimizers for γ big
- for $3 \leq N \leq 7$ and $\alpha \in (0, N-1)$ the ball is the unique global minimizer for γ small
- for all $N \geq 2$ and with $\alpha = N-2$ the ball is the unique global minimizer for γ small

Reference:

V. Julin,
Isoperimetric problem with a Coulombic repulsive term.
Indiana Univ. Math. J. (to appear).

The previous landscape (before BC)



Questions

- what happens for α small?
- can the ball be a global minimizer for big values of γ?
- non existence result for all $\alpha \in (0, N-1)$
- is the set of existence an interval?
- examples of global minimizers that are not balls
- how to deal with the case $\alpha \in [N-1, N)$?

Reference

Our second variation approach is based on the work:

E. Acerbi, N. Fusco, M. Morini,
Minimality via second variation for a nonlocal isoperimetric problem.
Comm. Math. Phys. **322** (2013), 515-557,

where they deal with the periodic case with a different nonlocal energy.

Variations

Let $X \in C^2(\mathbb{R}^N; \mathbb{R}^N)$ and let $\Phi : \mathbb{R}^N \times (-1,1) \to \mathbb{R}^N$ be its flow, *i.e.*

$$\frac{\partial \Phi}{\partial t} = X(\Phi)\,, \quad \Phi(x,0) := x\,.$$

Variations

Let $X \in C^2(\mathbb{R}^N; \mathbb{R}^N)$ and let $\Phi : \mathbb{R}^N \times (-1,1) \to \mathbb{R}^N$ be its flow, *i.e.*

$$\frac{\partial \Phi}{\partial t} = X(\Phi)\,, \quad \Phi(x,0) := x\,.$$

Define $E_t := \Phi(E,t)$ and suppose that X is s.t. $|E_t| = |E|$ for each t.

Variations

Let $X \in C^2(\mathbb{R}^N;\mathbb{R}^N)$ and let $\Phi : \mathbb{R}^N \times (-1,1) \to \mathbb{R}^N$ be its flow, *i.e.*

$$\frac{\partial \Phi}{\partial t} = X(\Phi)\,, \ \ \Phi(x,0) := x\,.$$

Define $E_t := \Phi(E,t)$ and suppose that X is s.t. $|E_t| = |E|$ for each t.
Thus consider

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(E_t)_{|_{t=0}} \qquad \text{and} \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{F}(E_t)_{|_{t=0}}$$

First variation **Second variation**

First variation

The first variation is given by

$$\frac{\mathrm{d}\mathcal{F}(E_t)}{\mathrm{d}t}_{|t=0} = \int_{\partial E} (H_{\partial E} + 2v_E)\langle X, \nu_E \rangle \,\mathrm{d}\mathcal{H}^{N-1} \,,$$

$H_{\partial E}$:= sum of the principal curvatures of ∂E, and $v_E(x) := \int_E \frac{1}{|x-y|^\alpha}\mathrm{d}y$.

First variation

The first variation is given by

$$\frac{\mathrm{d}\mathcal{F}(E_t)}{\mathrm{d}t}_{|t=0} = \int_{\partial E} (H_{\partial E} + 2v_E)\langle X, \nu_E\rangle \,\mathrm{d}\mathcal{H}^{N-1}\,,$$

$H_{\partial E} :=$ sum of the principal curvatures of ∂E, and $v_E(x) := \int_E \frac{1}{|x-y|^\alpha}\mathrm{d}y$.

Definition

We say that a regular set E is critical if

$$H_{\partial E} + 2v_E = \lambda\,, \qquad \text{on } \partial E\,,$$

where λ is the Lagrange multiplier given by the volume constraint.

Second variation

The second variation is of the form

$$\frac{\mathrm{d}^2\mathcal{F}(E_t)}{\mathrm{d}t^2}_{|t=0} = \partial^2\mathcal{F}(E)[\langle X, \nu_E\rangle] + R\,,$$

where R vanishes for critical sets.

Second variation

The second variation is of the form

$$\frac{\mathrm{d}^2\mathcal{F}(E_t)}{\mathrm{d}t^2}_{|t=0} = \partial^2\mathcal{F}(E)[\langle X, \nu_E\rangle] + R\,,$$

where R vanishes for critical sets.

Definition

We say that a regular critical set E is stable if

$$\partial^2\mathcal{F}(E)[\varphi] > 0\,, \qquad \text{for each } \varphi \in \widetilde{H}^1(\partial E) \setminus \{0\}\,.$$

Second variation

The second variation is of the form

$$\frac{\mathrm{d}^2\mathcal{F}(E_t)}{\mathrm{d}t^2}_{|_{t=0}} = \partial^2\mathcal{F}(E)[\langle X, \nu_E\rangle] + R\,,$$

where R vanishes for critical sets.

Definition

We say that a regular critical set E is stable if

$$\partial^2\mathcal{F}(E)[\varphi] > 0\,, \qquad \text{for each } \varphi \in \widetilde{H}^1(\partial E)\setminus\{0\}\,.$$

Note

Since

$$\partial^2\mathcal{F}(E)[\varphi] = \int_{\partial E}\int_{\partial E}\frac{\varphi(x)\varphi(y)}{|x-y|^\alpha}\,\mathrm{d}\mathcal{H}^{N-1}(x)\mathrm{d}\mathcal{H}^{N-1}(y) + \dots\,,$$

we need $\alpha \in (0, N-1)$ to have all well defined!

Necessary and sufficient (?) conditions

E is a local minimizer $\Longrightarrow$

E is critical

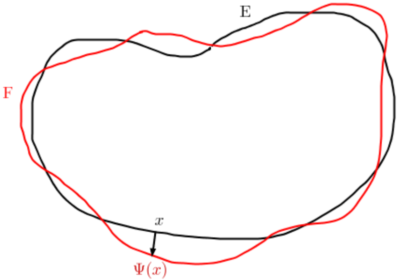
$\partial^2 \mathcal{F}(E)[\varphi] \geq 0$, for each $\varphi \in \widetilde{H}^1(\partial E)$

Necessary and sufficient (?) conditions

$$E \text{ is a local minimizer} \quad \Longrightarrow \quad \begin{cases} E \text{ is critical} \\ \partial^2\mathcal{F}(E)[\varphi] \geq 0, \text{ for each } \varphi \in \widetilde{H}^1(\partial E) \end{cases}$$

$$E \text{ is critical and stable} \quad \Longrightarrow \quad E \text{ is a local minimizer}$$

$W^{2,p}$-local minimality



We consider sets F whose boundary is given by

$$\partial F = \{x + \psi(x)\nu_E(x) \ : \ x \in \partial E\}\,,$$

where $\psi \in W^{2,p}(\partial E)$ and $p > \max\{2, N-1\}$.

$W^{2,p}$-local minimality

Let E be a regular critical and stable set.
Let F be a set that will be closed to E in $W^{2,p}$.

$W^{2,p}$-local minimality

Let E be a regular critical and stable set.
Let F be a set that will be closed to E in $W^{2,p}$.

$$\begin{array}{ccccc} \mathrm{E} & & \longrightarrow & & \mathrm{F} \\ E_0 & \to & E_t & \to & E_1 \end{array}$$

$W^{2,p}$-local minimality

Let E be a regular critical and stable set.
Let F be a set that will be closed to E in $W^{2,p}$.

$$\begin{array}{ccccc} \mathrm{E} & & \longrightarrow & & \mathrm{F} \\ E_0 & \to & E_t & \to & E_1 \end{array}$$

Define

$$g(t) := \mathcal{F}(E_t)\,.$$

Recalling that, by criticality of E, $g'(0) = 0$, we have

$$\mathcal{F}(F) - \mathcal{F}(E) = g(1) - g(0) = \int_0^1 (1-t) g''(t)\,\mathrm{d}t\,,$$

where

$$g''(t) = \frac{\mathrm{d}^2 \mathcal{F}(E_s)}{\mathrm{d}s^2}_{|s=t} = \partial^2 \mathcal{F}(E_t)[\langle X, \nu_{E_t}\rangle] + R_t$$

$W^{2,p}$-local minimality

Let E be a regular critical and stable set.
Let F be a set that will be closed to E in $W^{2,p}$.

$$\begin{array}{ccccc} \mathrm{E} & & \longrightarrow & & \mathrm{F} \\ E_0 & \to & E_t & \to & E_1 \end{array}$$

Define

$$g(t) := \mathcal{F}(E_t)\,.$$

Recalling that, by criticality of E, $g'(0) = 0$, we have

$$\mathcal{F}(F) - \mathcal{F}(E) = g(1) - g(0) = \int_0^1 (1-t) g''(t)\,\mathrm{d}t\,,$$

where

$$g''(t) = \frac{\mathrm{d}^2 \mathcal{F}(E_s)}{\mathrm{d}s^2}_{|_{s=t}} = \partial^2 \mathcal{F}(E_t)[\langle X, \nu_{E_t} \rangle] + R_t$$

$\Longrightarrow$ E is a $W^{2,p}$-isolated local minimizer

References

N. Fusco, M. Morini,
Equilibrium configurations of epitaxially strained elastic films: second order minimality conditions and qualitative properties of the solutions.
Arch. Rat. Mech. Anal. **203** (2012), 247-327.

M. Cicalese, G.P. Leonardi,
A selection principle for the sharp quantitative isoperimetric inequality.
Arch. Rat. Mech. Anal. **206** (2012), 617-643.

E. Acerbi, N. Fusco, M. Morini,
Minimality via second variation for a nonlocal isoperimetric problem.
Comm. Math. Phys. **322** (2013), 515-557.

Regularity results for Λ-minimizers

Definition of Λ-minimizer

We say that a set $E \subset \mathbb{R}^N$ is a Λ-*minimizer* of the area functional, for $\Lambda \geq 0$, if

$$\mathcal{P}(E) \leq \mathcal{P}(F) + \Lambda |E \triangle F| \,,$$

for each set $F \subset \mathbb{R}^N$.

Regularity results

Regularity results for Λ-minimizers are due to E. Gonzalez, U. Massari and I. Tamanini and are based on the ideas of De Giorgi for perimeter minimzers.

White Theorem

Theorem (White, 1994)

Let $E \subset \mathbb{R}^N$ be of class C^1 and let $E_n \subset \mathbb{R}^N$ with $\sup_n \mathcal{P}(E_n) < +\infty$ s.t.

White Theorem

Theorem (White, 1994)

Let $E \subset \mathbb{R}^N$ be of class C^1 and let $E_n \subset \mathbb{R}^N$ with $\sup_n \mathcal{P}(E_n) < +\infty$ s.t.

$$E_n \text{ are } \Lambda\text{-minimizers}$$

$$E_n \to E \text{ in } L^1(\mathbb{R}^N)$$

White Theorem

Theorem (White, 1994)

Let $E \subset \mathbb{R}^N$ be of class C^1 and let $E_n \subset \mathbb{R}^N$ with $\sup_n \mathcal{P}(E_n) < +\infty$ s.t.

$$E_n \text{ are } \Lambda\text{-minimizers}$$

$$E_n \to E \text{ in } L^1(\mathbb{R}^N)$$

Then for n large enough

$$E_n \text{ is of class } C^{1,\frac{1}{2}}$$

White Theorem

Theorem (White, 1994)

Let $E \subset \mathbb{R}^N$ be of class C^1 and let $E_n \subset \mathbb{R}^N$ with $\sup_n \mathcal{P}(E_n) < +\infty$ s.t.

$$E_n \text{ are } \Lambda\text{-minimizers}$$

$$E_n \to E \text{ in } L^1(\mathbb{R}^N)$$

Then for n large enough

$$E_n \text{ is of class } C^{1,\frac{1}{2}}$$

$$E_n \to E \text{ in } C^{1,\beta}(\partial E) \text{ for all } \beta \in \left(0, \frac{1}{2}\right),$$

i.e. $\partial E_n = \{x + \psi_n(x)\nu_E(x) \, : \, x \in \partial E\}$, with $\psi_n \to 0$ in $C^{1,\beta}(\partial E)$.

$W^{2,p}$-local minimality implies L^1-local minimality

Let E be a regular critical and stable set.

$W^{2,p}$-local minimality implies L^1-local minimality

Let E be a regular critical and stable set.
Suppose that there exists $(E_h)_h$ s.t.

$$\mathcal{F}(E_h) \leq \mathcal{F}(E)\,, \qquad F_h \stackrel{L^1}{\to} E\,.$$

$W^{2,p}$-local minimality implies L^1-local minimality

Let E be a regular critical and stable set.
Suppose that there exists $(E_h)_h$ s.t.

$$\mathcal{F}(E_h) \leq \mathcal{F}(E)\,, \qquad F_h \overset{L^1}{\to} E\,.$$

Then

$$E_h \rightsquigarrow F_h$$

where

$$\mathcal{F}(F_h) \leq \mathcal{F}(E)\,, \quad F_h \overset{L^1}{\to} E\,, \quad F_h \text{ are } \Lambda\text{-minimizers}\,.$$

$W^{2,p}$-local minimality implies L^1-local minimality

Let E be a regular critical and stable set.
Suppose that there exists $(E_h)_h$ s.t.

$$\mathcal{F}(E_h) \leq \mathcal{F}(E)\,, \qquad F_h \overset{L^1}{\to} E\,.$$

Then

$$E_h \rightsquigarrow F_h$$

where

$$\mathcal{F}(F_h) \leq \mathcal{F}(E)\,, \quad F_h \overset{L^1}{\to} E\,, \quad F_h \text{ are } \Lambda\text{-minimizers}\,.$$

Then

$$F_h \overset{L^1}{\to} E \quad \overset{\text{White Thm}}{\Longrightarrow} \quad F_h \overset{C^{1,\beta}}{\to} E$$

$W^{2,p}$-local minimality implies L^1-local minimality

Let E be a regular critical and stable set.
Suppose that there exists $(E_h)_h$ s.t.

$$\mathcal{F}(E_h) \leq \mathcal{F}(E)\,, \qquad F_h \stackrel{L^1}{\to} E\,.$$

Then

$$E_h \rightsquigarrow F_h$$

where

$$\mathcal{F}(F_h) \leq \mathcal{F}(E)\,, \quad F_h \stackrel{L^1}{\to} E\,, \quad F_h \text{ are } \Lambda\text{-minimizers}\,.$$

Then

$$F_h \stackrel{L^1}{\to} E \quad \stackrel{\text{White Thm}}{\Longrightarrow} \quad F_h \stackrel{C^{1,\beta}}{\to} E \quad \stackrel{\text{Eq. EL}}{\Longrightarrow} \quad F_h \stackrel{W^{2,p}}{\to} E$$

The main result

Main result - (Bonacini-Cristoferi, 2014)

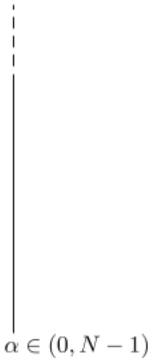
Assume that E is a regular critical and stable set for $\mathcal{F}$.
Then there exist $\delta > 0$ and $C > 0$ s.t.

$$\mathcal{F}(F) \geq \mathcal{F}(E) + C\big(\alpha(E,F)\big)^2$$

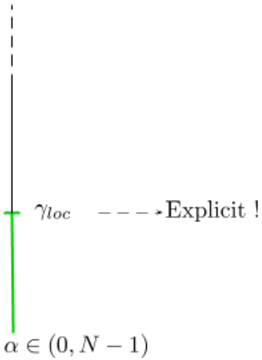
for every $F \subset \mathbb{R}^N$ such that $|F| = |E|$ and $\alpha(E,F) < \delta$, where

$$\alpha(E,F) := \min_{x \in \mathbb{R}^N} |E \triangle (F + x)|\,.$$

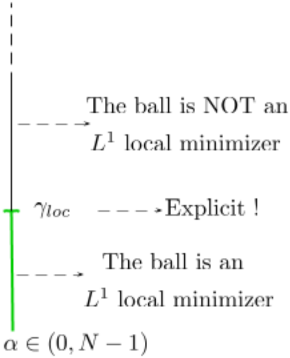
Local minimality of the ball - (Bonacini-Cristoferi, 2014)



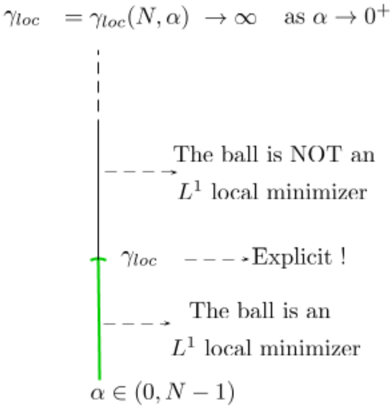
Local minimality of the ball - (Bonacini-Cristoferi, 2014)



Local minimality of the ball - (Bonacini-Cristoferi, 2014)



Local minimality of the ball - (Bonacini-Cristoferi, 2014)



Global minimality of the ball for γ small

Global minimality of the ball - (Bonacini-Cristoferi, 2014)

Given $N \geq 2$, $\alpha \in (0, N-1)$, let

$$\gamma_{\mathrm{glob}}(N,\alpha) := \sup\{\gamma \geq 0 : B_1 \text{ is a global minimizer for } \mathcal{F}_{\alpha,\gamma}\}\,.$$

Then:

- $\gamma_{\mathrm{glob}}(N,\alpha)$ is positive and finite
- the ball is a global minimizer of $\mathcal{F}_{\alpha,\gamma}$ if $\gamma \leq \gamma_{\mathrm{glob}}(N,\alpha)$
- it is the unique global minimizer of $\mathcal{F}_{\alpha,\gamma}$ if $\gamma < \gamma_{\mathrm{glob}}(N,\alpha)$

Global minimality of the ball for γ small

Suppose there exist $\gamma_n \to 0$ and $(E_n)_n$ with $|E_n| = |B_1|$ s.t.

$$\mathcal{F}_{\alpha,\gamma_n}(E_n) \leq \mathcal{F}_{\alpha,\gamma_n}(B_1)\,.$$

Global minimality of the ball for γ small

Suppose there exist $\gamma_n \to 0$ and $(E_n)_n$ with $|E_n| = |B_1|$ s.t.

$$\mathcal{F}_{\alpha,\gamma_n}(E_n) \leq \mathcal{F}_{\alpha,\gamma_n}(B_1)\,.$$

Then

$$\mathcal{P}(E_n) - \mathcal{P}(B_1) \leq \gamma_n\big(\mathcal{NL}_\alpha(B_1) - \mathcal{NL}_\alpha(E_n)\big)$$

Global minimality of the ball for γ small

Suppose there exist $\gamma_n \to 0$ and $(E_n)_n$ with $|E_n| = |B_1|$ s.t.

$$\mathcal{F}_{\alpha,\gamma_n}(E_n) \leq \mathcal{F}_{\alpha,\gamma_n}(B_1)\,.$$

Then

$$C\,|E_n \triangle B_1|^2 \leq \mathcal{P}(E_n) - \mathcal{P}(B_1) \leq \gamma_n\big(\mathcal{NL}_\alpha(B_1) - \mathcal{NL}_\alpha(E_n)\big)$$

Global minimality of the ball for γ small

Suppose there exist $\gamma_n \to 0$ and $(E_n)_n$ with $|E_n| = |B_1|$ s.t.

$$\mathcal{F}_{\alpha,\gamma_n}(E_n) \leq \mathcal{F}_{\alpha,\gamma_n}(B_1)\,.$$

Then

$$C\,|E_n \triangle B_1|^2 \leq \mathcal{P}(E_n) - \mathcal{P}(B_1) \leq \gamma_n(\mathcal{NL}_\alpha(B_1) - \mathcal{NL}_\alpha(E_n)) \leq \gamma_n c_0 |E_n \triangle B_1|$$

Global minimality of the ball for γ small

Suppose there exist $\gamma_n \to 0$ and $(E_n)_n$ with $|E_n| = |B_1|$ s.t.

$$\mathcal{F}_{\alpha,\gamma_n}(E_n) \leq \mathcal{F}_{\alpha,\gamma_n}(B_1)\,.$$

Then

$$C\,|E_n \triangle B_1|^2 \leq \mathcal{P}(E_n) - \mathcal{P}(B_1) \leq \gamma_n\big(\mathcal{NL}_\alpha(B_1) - \mathcal{NL}_\alpha(E_n)\big) \leq \gamma_n c_0 |E_n \triangle B_1|$$

Since $\gamma_n \to 0$:

$$E_n \to B_1 \quad \text{in } L^1\,.$$

Global minimality of the ball for α small

Global minimizers for α small - (Bonacini-Cristoferi, 2014)

There exists $\bar{\alpha} = \bar{\alpha}(N, \gamma) > 0$ such that for every $\alpha < \bar{\alpha}$

Global minimality of the ball for α small

Global minimizers for α small - (Bonacini-Cristoferi, 2014)

There exists $\bar{\alpha} = \bar{\alpha}(N, \gamma) > 0$ such that for every $\alpha < \bar{\alpha}$

- the ball is the unique global minimizer of $\mathcal{F}_{\alpha,\gamma}$ if $\gamma \leq \gamma_{\mathrm{glob}}(N, \alpha)$,

Global minimality of the ball for α small

Global minimizers for α small - (Bonacini-Cristoferi, 2014)

There exists $\bar{\alpha} = \bar{\alpha}(N, \gamma) > 0$ such that for every $\alpha < \bar{\alpha}$

- the ball is the unique global minimizer of $\mathcal{F}_{\alpha,\gamma}$ if $\gamma \leq \gamma_{\rm glob}(N, \alpha)$,
- for $\gamma > \gamma_{\rm glob}(N, \alpha)$ the minimum problem for $\mathcal{F}_{\alpha,\gamma}$ does not have a solution.

Global minimality of the ball for α small

Suppose there exist $\alpha_n \to 0$, $\gamma_n > 0$ and $(E_n)_n$ with $|E_n| = |B_1|$ which are not balls s.t.

$$\mathcal{F}_{\alpha_n,\gamma_n}(E_n) \leq \mathcal{F}_{\alpha_n,\gamma_n}(B_1)\,.$$

Global minimality of the ball for α small

Suppose there exist $\alpha_n \to 0$, $\gamma_n > 0$ and $(E_n)_n$ with $|E_n| = |B_1|$ which are not balls s.t.

$$\mathcal{F}_{\alpha_n,\gamma_n}(E_n) \leq \mathcal{F}_{\alpha_n,\gamma_n}(B_1)\,.$$

Then

$$\mathcal{P}(E_n) - \mathcal{P}(B_1) \leq \gamma_n(\mathcal{NL}_{\alpha_n}(B_1) - \mathcal{NL}_{\alpha_n}(E_n))$$

Global minimality of the ball for α small

Suppose there exist $\alpha_n \to 0$, $\gamma_n > 0$ and $(E_n)_n$ with $|E_n| = |B_1|$ which are not balls s.t.

$$\mathcal{F}_{\alpha_n,\gamma_n}(E_n) \leq \mathcal{F}_{\alpha_n,\gamma_n}(B_1)\,.$$

Then

$$C\,|E_n \triangle B_1|^2 \leq \mathcal{P}(E_n) - \mathcal{P}(B_1) \leq \gamma_n\big(\mathcal{NL}_{\alpha_n}(B_1) - \mathcal{NL}_{\alpha_n}(E_n)\big)$$

Global minimality of the ball for α small

Suppose there exist $\alpha_n \to 0$, $\gamma_n > 0$ and $(E_n)_n$ with $|E_n| = |B_1|$ which are not balls s.t.

$$\mathcal{F}_{\alpha_n,\gamma_n}(E_n) \leq \mathcal{F}_{\alpha_n,\gamma_n}(B_1)\,.$$

Then

$$C\,|E_n \triangle B_1|^2 \leq \mathcal{P}(E_n) - \mathcal{P}(B_1) \leq \gamma_n\big(\mathcal{NL}_{\alpha_n}(B_1) - \mathcal{NL}_{\alpha_n}(E_n)\big)$$

$$\leq \bar{\gamma}\big(\mathcal{NL}_{\alpha_n}(B_1) - \mathcal{NL}_{\alpha_n}(E_n)\big)$$

Global minimality of the ball for α small

Suppose there exist $\alpha_n \to 0$, $\gamma_n > 0$ and $(E_n)_n$ with $|E_n| = |B_1|$ which are not balls s.t.

$$\mathcal{F}_{\alpha_n,\gamma_n}(E_n) \leq \mathcal{F}_{\alpha_n,\gamma_n}(B_1)\,.$$

Then

$$C\,|E_n \triangle B_1|^2 \leq \mathcal{P}(E_n) - \mathcal{P}(B_1) \leq \gamma_n\big(\mathcal{NL}_{\alpha_n}(B_1) - \mathcal{NL}_{\alpha_n}(E_n)\big)$$

$$\leq \bar{\gamma}\big(\mathcal{NL}_{\alpha_n}(B_1) - \mathcal{NL}_{\alpha_n}(E_n)\big) \to 0$$

Global minimality of the ball for α small

Suppose there exist $\alpha_n \to 0$, $\gamma_n > 0$ and $(E_n)_n$ with $|E_n| = |B_1|$ which are not balls s.t.

$$\mathcal{F}_{\alpha_n,\gamma_n}(E_n) \leq \mathcal{F}_{\alpha_n,\gamma_n}(B_1)\,.$$

Then

$$C\,|E_n \triangle B_1|^2 \leq \mathcal{P}(E_n) - \mathcal{P}(B_1) \leq \gamma_n(\mathcal{NL}_{\alpha_n}(B_1) - \mathcal{NL}_{\alpha_n}(E_n))$$

$$\leq \bar{\gamma}(\mathcal{NL}_{\alpha_n}(B_1) - \mathcal{NL}_{\alpha_n}(E_n)) \to 0$$

Thus

$$E_n \to B_1 \text{ in } L^1\,.$$

Global minimality of the ball for α small

Suppose there exist $\alpha_n \to 0$, $\gamma_n > 0$ and $(E_n)_n$ with $|E_n| = |B_1|$ which are not balls s.t.

$$\mathcal{F}_{\alpha_n,\gamma_n}(E_n) \leq \mathcal{F}_{\alpha_n,\gamma_n}(B_1)\,.$$

Then

$$C\,|E_n \triangle B_1|^2 \leq \mathcal{P}(E_n) - \mathcal{P}(B_1) \leq \gamma_n(\mathcal{NL}_{\alpha_n}(B_1) - \mathcal{NL}_{\alpha_n}(E_n))$$

$$\leq \bar{\gamma}(\mathcal{NL}_{\alpha_n}(B_1) - \mathcal{NL}_{\alpha_n}(E_n)) \to 0$$

Thus

$$E_n \to B_1 \text{ in } L^1\,.$$

It can be proven that given $\bar{\alpha} > 0$ and $\bar{\gamma} > 0$ there exists $\delta > 0$ s.t.

$$\mathcal{F}_{\alpha,\gamma}(E) > \mathcal{F}_{\alpha,\gamma}(B_1)\,,$$

for every $\alpha \leq \bar{\alpha}$, $\gamma < \bar{\gamma}$ and each $|E| = |B_1|$ with $0 < \alpha(E, B_1) < \delta$.

Characterization of minimizing sequences for α small

Minimizing sequences for α small - (Bonacini-Cristoferi, 2014)

Let $\alpha < \bar{\alpha}$ and let

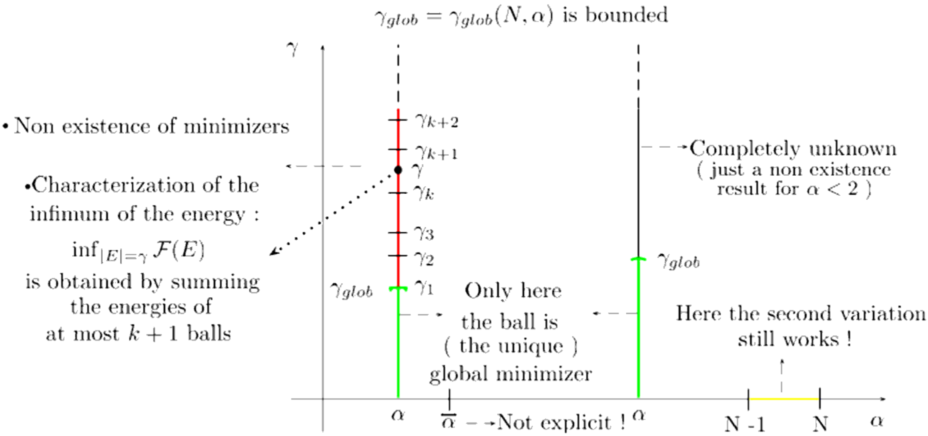
$$f_k(m) := \min_{\substack{\mu_1,\dots,\mu_k \geq 0 \\ \mu_1+\dots+\mu_k = m}} \left\{ \sum_{i=1}^{k} \mathcal{F}(B^i) : B^i \text{ ball}, \, |B^i| = \mu_i \right\}.$$

There exists an increasing sequence $(m_k)_k$, with $m_0 = 0$, $m_1 = \gamma_{\mathrm{glob}}$, such that $\lim_k m_k = \infty$ and

$$\inf_{|E|=m} \mathcal{F}(E) = f_k(m) \qquad \text{for every } m \in [m_{k-1}, m_k], \text{ for all } k \in \mathbb{N}, \tag{1}$$

that is, for every $m \in [m_{k-1}, m_k]$ a minimizing sequence for the total energy is obtained by a configuration of at most k disjoint balls with diverging mutual distance.
Moreover, the number of non-degenerate balls tends to $+\infty$ as $m \to +\infty$.

Global minimality results (after BC)



The case $\alpha \in [N-1, N)$

In the paper

A. Figalli, N. Fusco, F. Maggi, V. Millot, M. Morini,
Isoperimetry ans stability properties of balls with respect to nonlocal energies.
Preprint (2014).

they prove that for small masses the ball is the unique global minimizer.

Still open questions

- improve the characterization of minimizing sequences for α small
- non existence result for all $\alpha \in (0, N)$
- is the set of existence an interval?
- examples of global minimizers that are not balls

References

For all the details of this talk:

M. Bonacini, R.Cristoferi,
Local and global minimality results for a nonlocal isoperimetric problem in $\mathbb{R}^N$*.*
SIAM J. Math. Anal,. **46-4** (2014), 2310-2349.

Obrigado
pela
atenção !