Asymptotic spectral analysis in nanowires with axial heterogeneities

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Motivation

We consider the *stationary Schrödinger equation* with zero potential in a

thin bounded tubular domain of thickness δ : $\Omega_\delta \subset \mathbb{R}^3$

To find the spatial wave function ψ_δ and the energy levels E reduces to solve :

$$\begin{cases} -\frac{\hbar^2}{2m}\Delta\psi_\delta = E\,\psi_\delta, & \text{in } \Omega_\delta, \\ \psi_\delta = 0 & \text{on } \partial\Omega_\delta, \end{cases}$$

or, in a standard mathematical notation, forgetting the constants,

$$\begin{cases} -\Delta v_\delta = \lambda^\delta\, v_\delta, & \text{in } \Omega_\delta, \\ v_\delta = 0 & \text{on } \partial\Omega_\delta. \end{cases}$$

where $0 < \lambda_1^\delta \leq \lambda_2^\delta \leq \cdots \to +\infty$

Motivation

If the thin tube is non-homogeneous, the particle behaves as if its mass $m(x)$ varies and we are lead to consider the stationary Schrödinger effective-mass equation, with zero potential :

$$\begin{cases} -\frac{\hbar^2}{2}\,\mathrm{div}\,\left(\frac{1}{m(x)}\nabla\psi_\delta(x)\right) = E\,\psi_\delta(x), & x \in \Omega_\delta, \\ \psi_\delta(x) = 0 & x \in \partial\Omega_\delta, \end{cases}$$

or, following a more general notation $\left(A(x) = \frac{\hbar^2}{2\,m(x)}\mathbb{I}\right)$

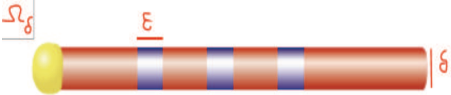
$$\begin{cases} -\mathrm{div}\,\left(A\nabla v_\delta\right) = \lambda^\delta\,v_\delta, & \text{in } \Omega_\delta, \\ v_\delta = 0 & \text{on } \partial\Omega_\delta. \end{cases}$$

where again $0 < \lambda_1^\delta \leq \lambda_2^\delta \leq \cdots \to +\infty$

- If the properties of the tube vary with high frequency, having ε-periodic heterogeneities, the energy levels will depend strongly on δ (or ε) and on the ratio δ/ε

Problem

$$\begin{cases} -\text{div}\,(A_\varepsilon \nabla v_\varepsilon^\delta) = \lambda_\varepsilon^\delta v_\varepsilon^\delta, & \text{in } \Omega_\delta, \\ v_\varepsilon^\delta \in H_0^1(\Omega_\delta) \end{cases}$$



$$0 < \lambda_\varepsilon^{\delta,0} \leqslant \lambda_\varepsilon^{\delta,1} \leqslant \cdots \leqslant \lambda_\varepsilon^{\delta,i} \leqslant \lambda_\varepsilon^{\delta,i+1} \leqslant \cdots$$

$$\delta \to 0 \;\Rightarrow\; \lambda_\varepsilon^{\delta,i} \to +\infty, \quad \forall i$$

- We analyse ε- periodic axial heterogeneities :
  $A_\varepsilon(x) = A(x_3/\varepsilon)$, A is [0,1] - periodic in $\mathbb{R}$
- We study the behavior of spectrum as $\varepsilon \to 0$ and $\delta \to 0$
- We look for the 1D limit problem

Notations and hypothesis

$\delta, \varepsilon > 0$ small parameters

$\Omega_\delta := \delta\omega \times I, \quad \omega \subset \mathbb{R}^2, \quad I := (0, L)$

$x^\delta = (\delta\bar{x}, x_3) \in \Omega_\delta, \quad \bar{x} = (x_1, x_2) \in \omega, \quad x_3 \in I, \quad Y := [0, 1]$

$u' := \dfrac{\partial u}{\partial x_3}, \qquad \bar{\Delta} u := \dfrac{\partial^2 u}{\partial x_1^2} + \dfrac{\partial^2 u}{\partial x_2^2}$

$A \in [L^\infty(\mathbb{R})]^{3\times 3}$, diagonal, Y-periodic,

$$A(y) = \text{ diag } \big(b(y), b(y), a(y)\big)$$

$$\eta \le a(y), b(y) \le \zeta,$$

for a.e. $y \in Y$ and for some $0 < \eta < \zeta$

$$A_\varepsilon(x_3) = \text{ diag } \big(b_\varepsilon(x_3), b_\varepsilon(x_3), a_\varepsilon(x_3)\big)$$

$$a_\varepsilon(x_3) = a\left(\frac{x_3}{\varepsilon}\right), \quad b_\varepsilon(x_3) = b\left(\frac{x_3}{\varepsilon}\right)$$

Rescaled problem

Usual procedure : change of variables, setting the problem in the fixed domain $\omega \times I$:

$$x^\delta = (\bar{x}^\delta, x_3) \in \Omega_\delta = \delta\omega \times I \mapsto x = \left(\frac{\bar{x}^\delta}{\delta}, x_3\right) \in \omega \times I$$

$$\tilde{v}_\varepsilon^\delta(x^\delta) = v_\varepsilon^\delta(x)$$

obtaining the rescaled problem :

$$\begin{cases} -\left(a_\varepsilon (v_\varepsilon^\delta)'\right)' - \frac{b_\varepsilon}{\delta^2} \bar{\Delta} v_\varepsilon^\delta = \lambda_\varepsilon^\delta \, v_\varepsilon^\delta, & \text{a.e. in } \omega \times I, \\ v_\varepsilon^\delta \in H_0^1(\omega \times I) \end{cases}$$

3 cases : $\delta = \varepsilon^\tau \quad \left(\delta/\varepsilon = \varepsilon^{(\tau-1)}\right)$

$\varepsilon \approx \delta : \quad \tau = 1$
$\varepsilon \ll \delta : \quad \tau < 1$
$\varepsilon \gg \delta : \quad \tau > 1$

Cellular and cross section problems

3 cases : $\varepsilon \approx \delta$: $\tau = 1$; $\varepsilon \ll \delta$: $\tau < 1$; $\varepsilon \gg \delta$: $\tau > 1$

We introduce the 1^{st} normalized eigenpair $(\mu^\tau_{\varepsilon,0}, \phi^\tau_{\varepsilon,0})$ of the 1D cell problem

$$\begin{cases} -\varepsilon^{2(\tau-1)}\Big(a(\phi^\tau_\varepsilon)'\Big)' + b\,\rho_0\,\phi^\tau_\varepsilon = \mu^\tau_\varepsilon \phi^\tau_\varepsilon, \text{ in } Y, \\ \phi^\tau_\varepsilon \in H^1_\#(Y), \end{cases}$$

where ρ_0 is the 1^{st} eigenvalue of the following 2D problem in the cross section ω :

$$\begin{cases} -\Delta\theta = \rho\,\theta, \text{ in } \omega, \\ \theta \in H^1_0(\omega), \end{cases}$$

$\mu^\tau_{\varepsilon,0}$ and ρ_0 are real, positive and simple, and the associated normalized eigenfunctions $\phi^\tau_{\varepsilon,0}$ and θ_0 may be chosen to be a strictly positive.

$\phi^\tau_{\varepsilon,0}$ is absolutely continuous.

Auxiliar results

Proposition 1

For each fixed τ, ε, we reduce the 3D eigenvalue pb to a family of 1D eigenvalue pbs, parametrized by the 2D eigenvalues in the cross section ω.

For n fixed, let $\{\lambda_k^{(n)}\}_{k\geq 0}$ be the k-sequence of eigenvalues for the 1D spectral pb :

$$\begin{cases} -\left(a\varphi'\right)' + b\,\rho_n\varphi = \lambda\,\varphi, \ \text{a.e. in } I, \\ \varphi \in H_0^1(I), \end{cases}$$

where $\{\rho_n\}$ is the sequence of eigenvalues of the 2D cross section pb.

Then, $\{\lambda_k^{(n)}\}_{k,n\geq 0}$ coincides with the sequence $\{\tilde{\lambda}_m\}$ of eigenvalues, repeated according to their multiplicity, of the 3D pb

$$\begin{cases} -\left(av'\right)' - b\bar{\Delta}v = \lambda v, \ \text{a.e. in } \omega \times I, \\ v \in H_0^1(\omega \times I). \end{cases}$$

The corresponding modes $\{v_k^{(n)}\}_{k,n\geq 0}$ are products $\theta_n(\bar{x})\,\varphi_k^{(n)}(x_3)$. □

Auxiliar results

Proposition 2

Fix $\tau, \varepsilon > 0$. Let $(\mu^\tau_{\varepsilon,0}, \phi^\tau_{\varepsilon,0})$ be the 1^{st} normalized eigenpair of the 1D cell pb. Relate v and u by the formula

$$v(x_3) = \phi^\tau_{\varepsilon,0}\left(\frac{x_3}{\varepsilon}\right)u(x_3), \text{ a.e. } x_3 \in I.$$

Then

- $v \in H^1_0(I) \iff u \in H^1_0(I)$
- $\int_I \left(a_\varepsilon\, |v'|^2 + \frac{b_\varepsilon}{\varepsilon^{2\tau}}\rho_0|v|^2 - \frac{\mu^\tau_{\varepsilon,0}}{\varepsilon^{2\tau}}|v|^2\right) dx_3 = \int_I \left(a_\varepsilon \left[\phi^\tau_{\varepsilon,0}\left(\frac{x_3}{\varepsilon}\right)\right]^2 |u'|^2\right) dx_3$
- $\lambda_{\varepsilon,k} = \frac{\mu^\tau_{\varepsilon,0}}{\varepsilon^{2\tau}} + \nu^\tau_{\varepsilon,k}$ ($\nu^\tau_{\varepsilon,k}$ are eigenvalues of right hand side pb)

Remark : The crucial point is to analyse the behavior of the 1^{st} eigenpair $(\mu^\tau_{\varepsilon,0}, \phi_{\varepsilon,0}\tau)$ of the cell pb, when $\tau \neq 1$ □

Auxiliar results

Proposition 3 (Vishik - Lyusternik)
Let $L : H \to H$ be a linear compact self-adjoint operator in a Hilbert space H, and suppose that $\lambda > 0$ and $f \in H$ are such that

$$\|Lf - \lambda f\|_H \leq \gamma.$$

Then

- There exists an eigenvalue $\bar{\lambda}$ of L :

$$|\bar{\lambda} - \lambda| \leq \gamma \|f\|_H^{-1}$$

- For $d > \gamma \|f\|_H^{-1}$, there exists a $\bar{f}$, $\|\bar{f}\|_H = \|f\|_H$, linear combination of eigenvectors associated with the eigenvalues lying in the interval $[\lambda - d, \lambda + d]$, satisfying

$$\|\bar{f} - f\|_H \leq 2\, d^{-1} \gamma$$

□

Case $\delta = \varepsilon$

- $\delta = \varepsilon \ (\tau = 1)$
- (μ_0, ϕ_0) is the 1^{st} normalized eigenpair of the cell pb :

$$\begin{cases} -\Big(a(y)\phi_0{}'\Big)' + b(y)\rho_0\,\phi_0 = \mu_0\phi_0, \ \text{ in } \ Y, \\ \phi_0 \in H^1_\#(Y) \end{cases}$$

  - $\mu_0 > 0$ simple
  - $\phi_0 > 0$, $\phi_0 \in H^1_\#(Y)$, continuous

- $\tilde{a}^h := \left(\dfrac{1}{|Y|} \displaystyle\int_Y \dfrac{1}{a(y)[\phi_0(y)]^2}\, dy \right)^{-1}$

Case $\delta = \varepsilon$

Theorem 1 $(\delta = \varepsilon)$

Let $(\lambda_{\varepsilon,k}, v_{\varepsilon,k})$ be the k^{th} eigenpair of the rescaled pb for $\delta = \varepsilon$.

Then

$$\lambda_{\varepsilon,k} = \frac{\mu_0}{\varepsilon^2} + \nu_{\varepsilon,k}$$

$$\nu_{\varepsilon,k} \to \nu_k$$

$$v_{\varepsilon,k} \rightharpoonup \theta_0(\bar{x})\ \varphi_k(x_3) \text{ in } L^2(\omega \times I)$$

where (ν_k, φ_k) is a k^{th} eigenpair :

$$\begin{cases} -\tilde{a}^h \varphi'' = \nu\varphi \\ \varphi \in H_0^1(I) \end{cases}$$

and conversely. □

Case $\delta \gg \varepsilon$

- $\delta = \varepsilon^\tau, \; 0 < \tau < 1$
- $(\mu^\tau_{\varepsilon,0}, \phi^\tau_{\varepsilon,0})$ normalized 1st eigenpair :

$$\begin{cases} -\varepsilon^{2(\tau-1)} \left(a(y)\, {\phi^\tau_{\varepsilon,0}}' \right)' + b(y)\, \rho_0\, \phi^\tau_{\varepsilon,0} = \mu^\tau_{\varepsilon,0}\, \phi^\tau_{\varepsilon,0}, \text{ in } Y, \\ \phi^\tau_{\varepsilon,0} \in H^1_\#(Y) \end{cases}$$

- $\psi_0 \equiv 1$ in Y, $\quad \gamma_j := \rho_0 \displaystyle\int_Y b(y)\, \psi_j(y)\, dy \;\; (j \in \mathbb{N}_0)$

  ψ_j solutions of the recurrence problems in $H^1_\#(Y)$

$$\begin{cases} -\left(a(y)\, {\psi_j}' \right)' = -b(y)\, \rho_0\, \psi_{j-1} + \sum_{k=0}^{j-1} \gamma_k\, \psi_{j-1-k}, \\ \int_Y \psi_j(y)\, dy = 0 \end{cases}$$

- $a^h := \left(\dfrac{1}{|Y|} \displaystyle\int_Y \dfrac{1}{a(y)}\, dy \right)^{-1}$

Case $\delta \gg \varepsilon$

Theorem 2 $(\delta \gg \varepsilon)$
Let $(\lambda_{\varepsilon,k}, v_{\varepsilon,k})$ be the k^{th} eigenpair of the rescaled pb and $i \in \mathbb{N}$: $\frac{i-1}{i} < \tau \leq \frac{i}{i+1}$.
Then, as $\varepsilon \to 0$,

$$\mu^{\tau}_{\varepsilon,0} \to \gamma_0 = \rho_0 \int_Y b(y)\ dy, \quad \phi^{\tau}_{\varepsilon,0}(\bar{x}/\varepsilon) \to \psi_0 \equiv 1 \text{ (uniformly)}$$

$$\lambda_{\varepsilon,k} = \frac{\mu^{\tau}_{\varepsilon,0}}{\varepsilon^{2\tau}} + \nu^{\tau}_{\varepsilon,k} = \sum_{j=0}^{i} \frac{\gamma_j}{\varepsilon^{2[\tau(j+1)-j]}} + \rho^{\tau}_{\varepsilon,i} + \nu^{\tau}_{\varepsilon,k}$$

- $\rho^{\tau}_{\varepsilon,i} \to 0, \quad \nu^{\tau}_{\varepsilon,k} \to {\nu_k}^{\tau},$
- $v_{\varepsilon,k} \to \theta_0(\bar{x})\, \varphi_k(x_3), \quad \text{in } L^2(\omega \times I)$
- (ν_k, φ_k) a k^{th} eigenpair :

$$\begin{cases} -a^h\, {\varphi_k}'' = \nu_k \varphi_k, \ \text{ in } \ I, \\ \varphi_k \in H^1_0(I). \end{cases}$$

□

Case $\delta \gg \varepsilon$

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$$\lambda_{\varepsilon,k} = \frac{\mu^\tau_{\varepsilon,0}}{\varepsilon^{2\tau}} + \nu^\tau_{\varepsilon,k} = \sum_{j=0}^{i} \frac{\gamma_j}{\varepsilon^{2[\tau(j+1)-j]}} + \rho^\tau_{\varepsilon,i} + \nu^\tau_{\varepsilon,k}$$

$$\tau = \tfrac{1}{2}, \quad \lambda_{\varepsilon,k} = \frac{\gamma_0}{\varepsilon} + \gamma_1 + \nu_{\varepsilon,k} + \rho^\tau_{\varepsilon,1};$$

- $\rho^\tau_{\varepsilon,i} \to 0, \quad \nu^\tau_{\varepsilon,k} \to {\nu_k}^\tau,$
- $v_{\varepsilon,k} \to \theta_0(\bar{x})\ \varphi_k(x_3), \quad \text{in } L^2(\omega \times I)$
- (ν_k, φ_k) a k^{th} eigenpair :

$$\begin{cases} -a^h\ \varphi_k'' = \nu_k \varphi_k, & \text{in } I, \\ \varphi_k \in H^1_0(I). \end{cases}$$

□

Theorem 2 $(\delta \gg \varepsilon)$

Let $(\lambda_{\varepsilon,k}, v_{\varepsilon,k})$ be the k^{th} eigenpair of the rescaled pb and $i \in \mathbb{N} : \frac{i-1}{i} < \tau \leq \frac{i}{i+1}$.
Then, as $\varepsilon \to 0$,

$$\mu^{\tau}_{\varepsilon,0} \to \gamma_0 = \rho_0 \int_Y b(y)\ dy, \quad \phi^{\tau}_{\varepsilon,0}(\bar{x}/\varepsilon) \to \psi_0 \equiv 1 \text{ (uniformly)}$$

$$\lambda_{\varepsilon,k} = \frac{\mu^{\tau}_{\varepsilon,0}}{\varepsilon^{2\tau}} + \nu^{\tau}_{\varepsilon,k} = \sum_{j=0}^{i} \frac{\gamma_j}{\varepsilon^{2[\tau(j+1)-j]}} + \rho^{\tau}_{\varepsilon,i} + \nu^{\tau}_{\varepsilon,k}$$

$$\tau = \tfrac{1}{2}, \quad \lambda_{\varepsilon,k} = \frac{\gamma_0}{\varepsilon} + \gamma_1 + \nu_{\varepsilon,k} + \rho^{\tau}_{\varepsilon,1};$$
$$\tau = \tfrac{2}{3}, \quad \lambda_{\varepsilon,k} = \frac{\gamma_0}{\varepsilon^{4/3}} + \frac{\gamma_1}{\varepsilon^{2/3}} + \gamma_2 + \nu_{\varepsilon,k} + \rho^{\tau}_{\varepsilon,2}$$

- $\rho^{\tau}_{\varepsilon,i} \to 0, \quad \nu^{\tau}_{\varepsilon,k} \to {\nu_k}^{\tau},$
- $v_{\varepsilon,k} \to \theta_0(\bar{x})\ \varphi_k(x_3), \quad \text{in } L^2(\omega \times I)$
- (ν_k, φ_k) a k^{th} eigenpair :

$$\begin{cases} -a^h\ \varphi_k'' = \nu_k \varphi_k, & \text{in } I, \\ \varphi_k \in H^1_0(I). \end{cases}$$

Case $\delta \gg \varepsilon$

Remark

If $\sum_{j \geqslant 0} \|\psi_j\|_{L^2(Y)} < +\infty$ then

$$\sum_{j\geq 0} |\gamma_j| < +\infty \qquad \sum_{j\geq 0} \gamma_j = \mu_0, \qquad \sum_{j\geq 0} \psi_j = \frac{\phi_0}{\int_Y \phi_0 \; d\bar{y}}.$$

((μ_0, ϕ_0) is the 1st eigenpair of the cell pb for $\tau = 1$)

Since $\frac{i-1}{i} < \tau \leqslant \frac{i}{i+1} < 1$:

$$\sum_{j=0}^{i} \frac{\gamma_j}{\varepsilon^{\tau(2j+2)-2j}} \; (\tau < 1) \to \frac{\mu_0}{\varepsilon^2} \; (\tau = 1)$$

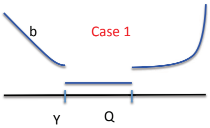
for fixed $\varepsilon > 0$ and as $\tau \to 1^-$ $(i \to +\infty)$.

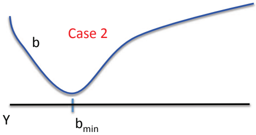
Cases $\delta \ll \varepsilon$

- $\delta = \varepsilon^\tau, \tau > 1$
- This case is more delicate to handle due to the degeneracy of the cell pb :

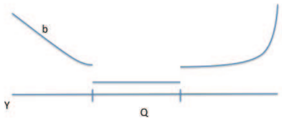
$$\begin{cases} -\varepsilon^{2(\tau-1)} \left(a(y)\, {\phi^\tau_{\varepsilon,0}}' \right)' + b(y)\, \rho_0\, \phi^\tau_{\varepsilon,0} = \mu^\tau_{\varepsilon,0}\, \phi^\tau_{\varepsilon,0}, \text{ in } Y, \\ \phi^\tau_{\varepsilon,0} \in H^1_\#(Y), \end{cases}$$

- The asymptotic behavior of $\mu^\tau_{\varepsilon,0}$ depends strongly on the behavior of the potential b.
- We will show two cases, for two different behaviors of b





$\delta \ll \varepsilon$ - case 1



- Let a be smooth
- $\exists Q$, open interval, $Q \subset\subset Y$:
  - $b \equiv b_{\min} := \min\limits_{Y} b$ on Q
  - b is smooth, $b > b_{\min}$ on $Y \backslash Q$
- Let (ν_0, q_0) be the normalized 1st eigenpair of pb

$$\begin{cases} -\left(a\, q'\right)' = \nu\, q, & \text{a.e. in } Q, \\ q \in H_0^1(Q) \end{cases}$$

$\delta \ll \varepsilon$ - case 1

Theorem 3 ($\delta \ll \varepsilon$ - case 1) Let $i \in \mathbb{N} : i \geq \frac{2}{\tau - 1}$. Then, as $\varepsilon \to 0^+$,

- $\mu^\tau_{\varepsilon,0} \to b_{\min}\, \rho_0\,, \quad \phi^\tau_{\varepsilon,0} \rightharpoonup q_0 \;\text{ in }\; H^1(Y)$
- $\lambda_{\varepsilon,0} = \dfrac{\mu^\tau_{\varepsilon,0}}{\varepsilon^{2\tau}} + \nu^\tau_{\varepsilon,k} = \dfrac{b_{\min}\, \rho_0}{\varepsilon^{2\tau}} + \dfrac{\nu_0}{\varepsilon^2} + \varepsilon^{\tau-3}\mu_1 + \cdots + \varepsilon^{i(\tau-1)-2}\mu_i + \rho^\tau_{\varepsilon,i} + \nu^\tau_{\varepsilon,0}$

$$|\rho^\tau_{\varepsilon,i}| \leqslant C\varepsilon^{(i+\frac{1}{2})\tau - (i+\frac{5}{2})} \to 0$$

$$\nu^\tau_{\varepsilon,0} := \inf_{\substack{\psi \in H^1_0(I) \\ \left\|\phi^\tau_{\varepsilon,0}\left(\frac{\cdot}{\varepsilon}\right)\psi\right\|_{L^2(I)}=1}} \left\{ \int_\omega \left|\phi^\tau_{\varepsilon,0}\left(\frac{x_3}{\varepsilon}\right)\right|^2 a\left(\frac{x_3}{\varepsilon}\right) \, |\psi'|^2 \; dx_3 \right\} \to 0^+$$

$$\phi^\tau_{\varepsilon,0}(y) = \varepsilon^{\tau-1} P_\varepsilon(y) + r_{i,\varepsilon}, \;\; y \in Y \backslash Q$$

$$\|r_{i,\varepsilon}\|_{L^2(Y\backslash Q)} \leq c_i\, \varepsilon^{2+\frac{1}{i}}\,; \quad |P_\varepsilon(y)|^2 \leqslant \textstyle\sum_{m=1}^{i} a_m \left(\frac{y}{\varepsilon}\right)^{j_m} e^{-b_m \frac{y}{\varepsilon}},$$

$a_m, b_m \in \mathbb{R}^+$ and $j_m \in \mathbb{N}$ independents of ε □

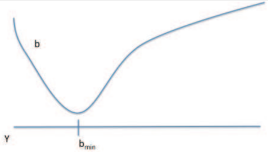
Remark

Thm is valid under weaker regularity hypotheses on the coefficients. It suffices that

- a are $C^{i+2}(Y)$,
- b is $C^{i+2}(Y \backslash Q)$,

  where i is the smallest natural number satisfying $i \geq \frac{2}{\tau - 1}$.

  The smaller $\tau - 1 > 0$ is, the more regularity is required



$\delta \ll \varepsilon$ - case 2

- Let b have a strict minimum in an interior point $y_0 \in Y$
  - $b(y_0) = b_{\min} := \min_Y b$
  - $b'(y_0) = 0$
  - $b''(y_0) > 0$
- Let a be smooth and b smooth in a neighborhood of y_0
- Let (ν_0, ψ_0) be the 1st normalized eigenpair of the harmonic oscillator

$$\begin{cases} -a(y_0)\, \psi'' + \dfrac{b''(y_0)\, \rho_0}{2}\, t^2\, \psi = \nu\, \psi, \text{ a.e. in } \mathbb{R}, \\ \psi \in H_0^1(\mathbb{R}) \end{cases}$$

$\delta \ll \varepsilon$ - case 2

Theorem 4 ($\delta \ll \varepsilon$ - case 2) Let $i \in \mathbb{N} : i \geq \frac{\tau+3}{\tau-1}$. Then, as $\varepsilon \to 0^+$,

- $\mu^\tau_{\varepsilon,0} \to b_{\min}\, \rho_0$
- $\lambda_{\varepsilon,0} = \dfrac{\mu^\tau_{\varepsilon,0}}{\varepsilon^{2\tau}} + \nu^\tau_{\varepsilon,k} =$

$$\frac{b_{\min}\, \rho_0}{\varepsilon^{2\tau}} + \frac{\nu_0}{\varepsilon^{\tau+1}} + \frac{\mu_3}{\varepsilon^{\frac{\tau+3}{2}}} + \cdots + \varepsilon^{\frac{i+2}{2}(\tau-1)-2\tau}\mu_{i+2} + \rho^\tau_{\varepsilon,i} + \nu^\tau_{\varepsilon,0}$$

$$|\rho^\tau_{\varepsilon,i}| \leq C\, \varepsilon^{\frac{(i+3)\,(\tau-1)}{4\,\tau}} \to 0$$

$$\nu^\tau_{\varepsilon,0} := \inf_{\substack{\psi \in H^1_0(I) \\ \left\|\phi^\tau_{\varepsilon,0}\left(\frac{\cdot}{\varepsilon}\right)\psi\right\|_{L^2(I)}=1}} \left\{ \int_\omega \left|\phi^\tau_{\varepsilon,0}\left(\frac{x_3}{\varepsilon}\right)\right|^2 a\left(\frac{x_3}{\varepsilon}\right)\ |\psi'|^2\ dx_3 \right\} \to 0^+$$

$$\phi^\tau_{\varepsilon,0}(y) = \varepsilon^{-(\tau-1)/4}\, \psi_0\left(\frac{y-y_0}{\varepsilon^{(\tau-1)/2}}\right) \sigma_\varepsilon(y) + R^\tau_{\varepsilon,0}$$

$$\|R^\tau_{\varepsilon,0}\|_{L^2(Y)} \leq c\, \varepsilon^{(\tau-1)/2}$$

(ν_0, ψ_0) is the 1^{st} normalized eigenpair of the harmonic oscillator

σ_ε is a convenient cut-off function in Y. □

Main steps

- Using Proposition 1 we turn the rescaled problem for $\delta = \varepsilon^\tau$:

$$\begin{cases} -(a_\varepsilon v_\varepsilon')' - \frac{b_\varepsilon}{\varepsilon^{2\tau}} \bar{\Delta} v_\varepsilon = \lambda_\varepsilon^\delta v_\varepsilon, & \text{a.e. in } \omega \times I, \\ v_\varepsilon \in H_0^1(\omega \times I) \end{cases}$$

into the 1D pb

$$\begin{cases} -(a_\varepsilon v_\varepsilon')' + \frac{b_\varepsilon \ \rho_0}{\varepsilon^{2\tau}} v_\varepsilon = \lambda_\varepsilon^\delta v_\varepsilon, & \text{a.e. in } I, \\ v_\varepsilon \in H_0^1(I) \end{cases}$$

- Using Proposition 2 we obtain that :

$$\int_I \left(a_\varepsilon \, |v'|^2 + \frac{b_\varepsilon}{\varepsilon^{2\tau}} \rho_0 |v|^2 - \frac{\mu_{\varepsilon,0}^\tau}{\varepsilon^{2\tau}} v \right) dx = \int_I \left(a_\varepsilon \left[\phi_{\varepsilon,0}^\tau \left(\frac{x_3}{\varepsilon} \right) \right]^2 |u'|^2 \right) dx$$

$\lambda_{\varepsilon,k} = \dfrac{\mu_{\varepsilon,0}^\tau}{\varepsilon^{2\tau}} + \nu_{\varepsilon,k}^\tau$ $\qquad$ ($\nu_{\varepsilon,k}^\tau$ are eigenvalues of right hand side pb.)

- The crucial point is to analyse the behavior of the 1^{st} eigenpair $(\mu^{\tau}_{\varepsilon,0}, \phi_{\varepsilon,0}\tau)$ of the cell pb, when $\tau \neq 1$:

$$\begin{cases} -\varepsilon^{2(\tau-1)}\left(a(y)\,{\phi^{\tau}_{\varepsilon,0}}'\right)' + b(y)\,\rho_0\,\phi^{\tau}_{\varepsilon,0} = \mu^{\tau}_{\varepsilon,0}\,\phi^{\tau}_{\varepsilon,0}, & \text{in } Y, \\ \phi^{\tau}_{\varepsilon,0} \in H^1_{\#}(Y) \end{cases}$$

- We use classical homogenization results, formal expansions and Proposition 3, to justify the development of the eigenpair
- The development must attain an order superior to 2τ
- In the degenerate cases $\tau > 1$ we must also use convenient changes of variables, depending on de behavior of the potencial b.

Idea of the proofs

$\tau > 1$- case 2 :

- For the particular case a constant and $b(y) = b_{\min} + c(y - 1/2)^2$, the cell pb will behave like the quantum harmonic oscillator :

$$-a(\phi^\tau_{\varepsilon,0})'' + \frac{c\,\rho_0}{\varepsilon^{2(\tau-1)}}\,(y - 1/2)^2\,\phi^\tau_{\varepsilon,0} = \left(\frac{\mu^\tau_{\varepsilon,0} - b_{\min}\,\rho_0}{\varepsilon^{2(\tau-1)}}\right)\,\phi^\tau_{\varepsilon,0}$$

  This give us a hint of how to analyze the general case.

- When the coefficient $\nu_0 = \sqrt{\frac{c\,\rho_0}{\varepsilon^{2(\tau-1)}}}$ is of order $1/\varepsilon^{(\tau-1)}$, then the corresponding principal mode ψ_0 behaves like a function of $t/\sqrt{\varepsilon^{\tau-1}}$, divided by $\sqrt[4]{\varepsilon^{\tau-1}}$.

  This justifies the following change of variables, forgetting the fixed index τ :

$$\epsilon = \sqrt{\varepsilon^{\tau-1}},\ \ t = \frac{y - y_0}{\epsilon},\ \ \varphi_\epsilon(t) = \phi^\tau_\varepsilon(y_0 + \epsilon t),\ \ t \in \left[\frac{-y_0}{\epsilon}, \frac{1 - y_0}{\epsilon}\right]$$

Idea of the proofs

$\tau > 1$- case 2 :

- Cell pb becomes :

$$\begin{cases} -\epsilon^2 \Big(a(y_0+\epsilon t)(\varphi_\epsilon)' \Big)' + b(y_0+\epsilon t)\, \rho_0\, \varphi_\epsilon = \mu_\epsilon \varphi_\epsilon, \\ \varphi_\epsilon \in H^1_\# \left(\frac{-y_0}{\epsilon}, \frac{1-y_0}{\epsilon} \right). \end{cases}$$

- We assume the expansions :

$$\mu_\epsilon = b_{\min}\, \rho_0 + \epsilon^2 \mu_2 + \epsilon^3 \mu_3 + \cdots + \epsilon^k \mu_k + \cdots$$

$$\varphi_\epsilon(t) = \varphi_0(t) + \epsilon \varphi_1(t) + \epsilon^2 \varphi_2(t) + \cdots + \epsilon^{k-2} \varphi_{k-2}(t) + \cdots$$

$$a(y_0+\epsilon t) = a(y_0) + \epsilon a'(y_0) t + \epsilon^2 \frac{a''(y_0)}{2} t^2 + \cdots + \epsilon^{k-1} \frac{a^{(k-1)}(y_0)}{(k-1)!} t^{k-1} + \epsilon^k \frac{a^{(k)}(\theta)}{(k)!} t^k$$

$$b(y_0+\epsilon t) = b_{\min} + \epsilon^2 \frac{b''(y_0)}{2} t^2 + \epsilon^3 \frac{b'''(y_0)}{3!} t^3 + \cdots \epsilon^{k-1} \frac{b^{(k-1)}(y_0)}{(k-1)!} t^{k-1} + \epsilon^k \frac{b^{(k)}(\xi)}{(k)!} t^k$$

- We apply the usual asymptotic techniques.

□

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