Phase field approximation of cohesive fracture models

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joint work with

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Introduction

Typical variational antiplane shear models in Fracture Mechanics take the form

$$\int_\Omega h(|\nabla u|)dx + \int_{J_u} g(|[u]|)d\mathcal{H}^{n-1} + \kappa |D^c u|(\Omega),$$

for a scalar displacement $u \in BV(\Omega)$. Here

- strain energy $\rightsquigarrow$ h is quadratic near the origin;
- surface energy $\rightsquigarrow$ g is nondecreasing;
- micro-cracking $\rightsquigarrow$ $\kappa \in [0, +\infty]$ is the slope of g at 0 and of h at $+\infty$.

Examples:

- Griffith fracture $\rightsquigarrow$ $h(s) = s^2$, g is constant, and $\kappa = +\infty$;
- Barenblatt cohesive fracture $\rightsquigarrow$ $\kappa \in (0, +\infty)$, h is linear at $+\infty$, g is concave, growing from $g(0) = 0$ to $g(+\infty) < +\infty$, and is linear at 0;
- Dugdale cohesive fracture $\rightsquigarrow$ as before with $g(s) = s \wedge 1$;

Regularization *via* Γ-convergence

A large literature was devoted to the derivation of brittle fracture models

$$\int_\Omega |\nabla u|^2 dx + \mathcal{H}^{n-1}(J_u), \qquad u \in SBV(\Omega)$$

from more regular models, like damage or phase field models, mainly within the framework of Γ-convergence:

- antiplane shear ⇝ [Ambrosio–Tortorelli '90, '92];
- vector-valued case ⇝ [Focardi '01];
- linearized elasticity ⇝ [Chambolle '04, '05], [I. '13];
- numerical simulations ⇝ [Bellettini–Coscia '94], [Bourdin–Francfort–Marigo '00], [Burke–Ortner–Süli '10]. . . ;
- . . .

A smaller amount of literature looks towards cohesive fracture:

- [Ambrosio–Lemenant–Royer-Carfagni '13], [Dal Maso–I. '13], [I. '13], [Focardi–I. '14] for the functional

$$\int_\Omega |\nabla u|^2 dx + \int_{J_u} |[u]| d\mathcal{H}^{n-1} + \mathcal{H}^{n-1}(J_u), \qquad u \in SBV(\Omega);$$

- [Alicandro–Braides–Shah '99], [Alicandro–Focardi '02] for the functional

$$\int_\Omega |\nabla u| dx + \int_{J_u} g(|[u]|) d\mathcal{H}^{n-1} + \kappa |D^c u|(\Omega), \qquad u \in BV(\Omega);$$

- [I. '13] for the functional

$$\int_\Omega h(|\nabla u|) dx + \int_{J_u} |[u]| d\mathcal{H}^{n-1} + |D^c u|(\Omega), \qquad u \in BV(\Omega).$$

Goal: to approximate *via* Γ-convergence the cohesive energy functional à la Barenblatt.

The regularized functionals

We study a damage model as proposed by [Pham–Marigo '10], namely,

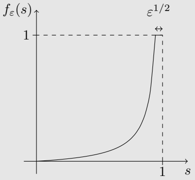
$$F_\varepsilon(u,v) := \int_\Omega \left(f_\varepsilon^2(v)|\nabla u|^2 + \frac{(1-v)^2}{4\varepsilon} + \varepsilon|\nabla v|^2 \right) dx,$$

with $u, v \in H^1(\Omega)$, $0 \le v \le 1$ $\mathcal{L}^n$-a.e. in Ω, and $F_\varepsilon(u,v) := +\infty$ otherwise. The potential $f_\varepsilon : [0,1) \to [0,+\infty]$ is defined by

$$f_\varepsilon(s) := 1 \wedge \varepsilon^{1/2} f(s),$$

where $f \in C^0([0,1),[0,+\infty))$ is nondecreasing, $f^{-1}(0) = \{0\}$, and it satisfies $\lim_{s\to 1}(1-s)f(s) = \ell, \quad \ell \in (0,+\infty)$.

Prototypical case: $f(s) = \dfrac{s}{1-s}$.



Theorem

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz set. Then, the functionals F_ε Γ-converge in $L^1(\Omega)\times L^1(\Omega)$ to the functional F defined by

$$F(u,v) := \begin{cases} \int_\Omega h(|\nabla u|)dx + \int_{J_u} g(|[u]|)d\mathcal{H}^{n-1} + \ell|D^c u|(\Omega) & \text{if } v = 1 \ \mathcal{L}^n\text{-a.e.,} \\ & u \in GBV(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Here the volume energy density h is set as $h(s) := s^2$ if $s \le \ell/2$ and as $h(s) := \ell s - \ell^2/4$ otherwise, while the surface energy density g is given by

$$g(s) := \inf\left\{ \int_0^1 |1-\beta|\sqrt{f^2(\beta)|\alpha'|^2 + |\beta'|^2}\,dt : \ (\alpha,\beta) \in H^1\big((0,1)\big), \right.$$
$$\left. \alpha(0) = 0,\ \alpha(1) = s,\ \beta(0) = \beta(1) = 1 \right\}.$$

Convergence of minima

Let $0 < \eta_\varepsilon, \varepsilon$ be such that $\eta_\varepsilon = o(\varepsilon)$ and let $\zeta \in L^q(\Omega)$, with $q > 1$. Let $G_\varepsilon \colon L^1(\Omega) \times L^1(\Omega) \to [0, +\infty]$ be defined by

$$G_\varepsilon(u, v) := \int_\Omega \left(\left(f_\varepsilon^2(v) + \eta_\varepsilon\right)|\nabla u|^2 + \frac{(1-v)^2}{4\varepsilon} + \varepsilon|\nabla v|^2 + |u - \zeta|^q \right) dx$$

if $(u, v) \in H^1(\Omega) \times H^1(\Omega)$ and $0 \le v \le 1$ $\mathcal{L}^n$-a.e. in Ω, and $+\infty$ otherwise. Let now $\mathcal{G} \colon L^1(\Omega) \to [0, +\infty]$ be defined by

$$\mathcal{G}(u) := \int_\Omega h(|\nabla u|)dx + \int_{J_u} g(|[u]|)d\mathcal{H}^{n-1} + \ell|D^c u|(\Omega) + \int_\Omega |u - \zeta|^q dx$$

if $u \in GBV(\Omega)$ and $+\infty$ otherwise.

Corollary

Let $(u_\varepsilon, v_\varepsilon) \in H^1(\Omega) \times H^1(\Omega)$ be a minimizer of G_ε. Then $v_\varepsilon \to 1$ in $L^1(\Omega)$, a subsequence of u_ε converges in $L^q(\Omega)$ to a minimizer u of $\mathcal{G}$, and the minimum values of G_ε tend to the minimum value of $\mathcal{G}$.

Remarks

Recalling that

$$F_\varepsilon(u,v) := \int_\Omega \left(f_\varepsilon^2(v)|\nabla u|^2 + \frac{(1-v)^2}{4\varepsilon} + \varepsilon|\nabla v|^2 \right) dx,$$

if $u, v \in H^1(\Omega)$ and $0 \le v \le 1$ $\mathcal{L}^n$-a.e. in Ω, where $f_\varepsilon(s) := 1 \wedge \varepsilon^{1/2} f(s)$, we observe that

- the truncation of f can be performed with any continuous nondecreasing function ψ satisfying $\inf \psi > 0$ and $\psi(1) = 1$, therefore f_ε can be made increasing.

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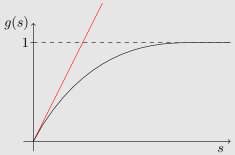
- In the Ambrosio–Tortorelli functional we have $f_\varepsilon^2(v) = v^2$, so there are essentially two regimes: $v \sim 0$ and $v \sim 1$. Moreover you can separate the contributions. In our case it can be $v \sim 1$ and $f_\varepsilon(v) = \varepsilon^{1/2} f(v)$.

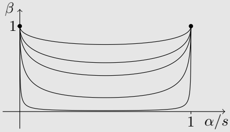
By $\lim_{s\to 1}(1-s)f(s)=\ell$, $\ell\in(0,+\infty)$, $f^{-1}(0)=0$, and

$$g(s):=\inf\left\{\int_0^1 |1-\beta|\sqrt{f^2(\beta)|\alpha'|^2+|\beta'|^2}\,dt:\ (\alpha,\beta)\in H^1\big((0,1)\big),\right.$$
$$\left.\alpha(0)=0,\ \alpha(1)=s,\ \beta(0)=\beta(1)=1\right\}$$

we observe that

- g is nondecreasing, subadditive, Lipschitz, linear at 0, satisfies $g(0)=0$, and tends to 1 as $s\to+\infty$.
  In particular when $s\sim 0$ the optimal (α,β) is near to $(st,1)$; when $s\sim+\infty$, it is near to the Ambrosio–Tortorelli optimal curve.





By $\lim_{s\to 1}(1-s)f(s) = \ell$, $\ell \in (0,+\infty)$, $f^{-1}(0) = 0$, and

$$g(s) := \inf\left\{\int_0^1 |1-\beta|\sqrt{f^2(\beta)|\alpha'|^2 + |\beta'|^2}\,dt : \ (\alpha,\beta) \in H^1\big((0,1)\big),\right.$$
$$\left.\alpha(0) = 0,\ \alpha(1) = s,\ \beta(0) = \beta(1) = 1\right\}$$

we observe that

- the set of all possible g's is large:
  - if $\ell_j = \ell$ for all j, $f^{(j)} \geq f^{(j+1)}$, and $f^{(j)}(s) \downarrow 0$ for all $s \in [0,1)$, then $g_j \geq g_{j+1}$ and $g_j(s) \downarrow 0$ for all $s \in [0,+\infty)$;
  - if $\ell_j = \ell$ for all j, $f^{(j)} \leq f^{(j+1)}$, and $f^{(j)}(s) \uparrow +\infty$ for all $s \in (0,1)$, then $g_j \leq g_{j+1}$ and $g_j(s) \uparrow 1 \wedge \ell s$ for all $s \in [0,+\infty)$.

Further results: recovering Griffith's brittle fracture

We consider a sequence of nondecreasing continuous functions $(f^{(j)})$ with $f^{(j)^{-1}}(0) = \{0\}$, and $\lim_{s\to 1}(1-s)f^{(j)}(s) = \ell_j$, $\ell_j \in (0,+\infty)$. For all $j,k \in \mathbb{N}$

$$F_k^{(j)}(u,v) := \int_\Omega \Big((f_k^{(j)})^2(v)|\nabla u|^2 + \frac{(1-v)^2}{4\varepsilon_k} + \varepsilon_k|\nabla v|^2 \Big) dx$$

if $(u,v) \in H^1(\Omega)\times H^1(\Omega)$ and $0 \le v \le 1$ $\mathcal{L}^n$-a.e. in Ω, and $+\infty$ otherwise, where $f_k^{(j)}(s) := 1 \wedge \varepsilon_k^{1/2} f^{(j)}(s)$.

Theorem

Suppose that $(f^{(j)})$ satisfies $f^{(j)} \le f^{(j+1)}$, $\ell_j \uparrow +\infty$ and $f^{(j)}(s) \uparrow \infty$ pointwise in $(0,1)$. Then, the functionals $F_k^{(k)}$ Γ-converge in $L^1(\Omega)\times L^1(\Omega)$ to

$$\widetilde{MS}(u,v) := \int_\Omega |\nabla u|^2 dx + \mathcal{H}^{n-1}(J_u)$$

if $u \in GSBV(\Omega)$, $v = 1$ $\mathcal{L}^n$-a.e. in Ω, and $+\infty$ otherwise.

Further results: Dugdale's cohesive model

We consider $f^{(j)}(s) := (a_j s) \vee f(s)$, with f as before and (a_j) nondecreasing, $a_j \uparrow +\infty$, such that $a_j \varepsilon_j^{1/2} \downarrow 0$. Let $f_k^{(j)}(s) := 1 \wedge \varepsilon_k^{1/2} f^{(j)}(s)$ and let

$$F_k^{(j)}(u,v) := \int_\Omega \left((f_k^{(j)})^2(v)|\nabla u|^2 + \frac{(1-v)^2}{4\varepsilon_k} + \varepsilon_k |\nabla v|^2 \right) dx$$

if $(u,v) \in H^1(\Omega) \times H^1(\Omega)$ and $0 \leq v \leq 1$ $\mathcal{L}^n$-a.e. in Ω, and $+\infty$ otherwise.

Theorem

The functionals $F_k^{(k)}$ Γ-converge in $L^1(\Omega) \times L^1(\Omega)$ to

$$\widetilde{\mathfrak{D}}(u,v) := \int_\Omega h(|\nabla u|) dx + \int_{J_u} \left(1 \wedge \ell |[u]|\right) d\mathcal{H}^{n-1} + \ell |D^c u|(\Omega)$$

if $v = 1$ $\mathcal{L}^n$-a.e. in Ω, and $+\infty$ otherwise. Here $h(s) := s^2$ if $s \leq \ell/2$ and as $h(s) := \ell s - \ell^2/4$ otherwise.

Further results: a model with power-law growth

Let $p > 1$ and consider a nondecreasing continuous function ψ_p satisfying ${\psi_p}^{-1}(0) = 0$ and

$$\lim_{s\to 1}(1-s)^p\psi_p(s) = \kappa, \qquad \kappa \in (0,+\infty),$$

the prototype being $\psi_p(s) := s/(1-s)^p$. Let

$$f^{(j)}(s) := \frac{j\,s}{1-s} \wedge \psi_p(s),$$

let $f_k^{(j)}(s) := 1 \wedge \varepsilon_k^{1/2} f^{(j)}(s)$, and let

$$F_k^{(j)}(u,v) := \int_\Omega \left((f_k^{(j)})^2(v)|\nabla u|^2 + \frac{(1-v)^2}{4\varepsilon_k} + \varepsilon_k|\nabla v|^2 \right) dx$$

if $(u,v) \in H^1(\Omega)\times H^1(\Omega)$ and $0 \le v \le 1$ $\mathcal{L}^n$-a.e. in Ω, and $+\infty$ otherwise.

Theorem

The functionals $F_k^{(k)}$ Γ*-converge in* $L^1(\Omega)\times L^1(\Omega)$ *to* $\widetilde{\Phi}_p$*, where*

$$\widetilde{\Phi}_p(u,v) := \int_\Omega |\nabla u|^2 dx + \int_{J_u} \vartheta_p(|[u]|)\,d\mathcal{H}^{n-1}$$

if $u \in GSBV(\Omega)$ *and* $v = 1$ $\mathcal{L}^n$*-a.e. in* Ω, $+\infty$ *otherwise. Here*

$$\vartheta_p(s) := \inf\left\{\int_0^1 |1-\beta|\sqrt{\psi_p^2(\beta)|\alpha'|^2 + |\beta'|^2}\,dt : (\alpha,\beta)\in H^1\big((0,1)\big), \right.$$
$$\left. \alpha(0)=0,\ \alpha(1)=s,\ \beta(0)=\beta(1)=1\right\}.$$

Note that ϑ_p is nondecreasing, subadditive, $\frac{2}{p+1}$-Hölder, satisfies $\vartheta_p(0)=0$, grows as $s^{\frac{2}{p+1}}$ near 0, and tends to 1 as $s\to+\infty$.

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...thanks for your attention!